# The correlated bivariate noncentral F distribution with application

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# The Correlated Bivariate Noncentral *F* Distribution and Its Application

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*This article proposes the* singly *and* doubly *correlated bivariate noncentral* F *(BNCF) distributions. The probability density function (pdf) and the cumulative distribution function (cdf) of the distributions are derived for arbitrary values of the parameters. The pdf and cdf of the distributions for different arbitrary values of the parameters are computed, and their graphs are plotted by writing and implementing new R codes. An application of the correlated BNCF distribution is illustrated in the computations of the power function of the pre-test test for the multivariate simple regression model (MSRM).*

**Keywords** Bivariate noncentral chi-square distribution; Compounding distribution; Correlated bivariate noncentral *F* distribution; Noncentrality parameter; Power function; Pre-test test

**Mathematics Subject Classification** Primary 62H10; Secondary 60E05

## 1. Introduction

The bivariate central *F* (BCF) distribution has been studied by many authors, including Krishnaiah (1965a), Amos and Bulgren (1972), Schuurmann et al. (1975), Johnson et al. (1995), and El-Bassiouny and Jones (2009). Krishnaiah (1965b) described the use of the BCF distribution in a problem of simultaneous statistical inference. Krishnaiah and Armitage (1965) later studied the multivariate central *F* distribution. Hewett and Bulgren (1971) studied the prediction interval for failure times in certain life testing experiments using the multivariate central *F* distribution.

Many authors have also studied the univariate noncentral *F* distribution, including Mudholkar et al. (1976), Muirhead (1982), Johnson et al. (1995), and Shao (2005). Johnson

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et al. (1995) provided the definition of the univariate noncentral $F$ distribution known as the *singly* noncentral $F$ distribution. The authors also described the *doubly* noncentral $F$ distribution with $(\nu_1, \nu_2)$ degrees of freedom and noncentrality parameters $\lambda_1$ and $\lambda_2$ as the ratio of two independent noncentral chi-square variables, $\chi_{\nu_1}^{\prime 2}(\lambda_1)/\nu_1$ and $\chi_{\nu_2}^{\prime 2}(\lambda_2)/\nu_2$. Tiku (1966) proposed an approximation to the multivariate noncentral $F$ distribution.

In the study of improving the power of a statistical test by pre-testing the uncertain nonsample prior information (NSPI) on the value of a set of parameters (see Saleh and Sen, 1983; Yunus and Khan, 2011a), the cdf of a bivariate noncentral chi-square distribution is used to compute the power function of the test. For large sample studies, the cdf of the bivariate noncentral chi-square (BNCC) distribution is used to compute the power function of the test for testing one subset of regression parameters after pre-testing on another subset of parameters of a multivariate simple regression model (MSRM; see Saleh and Sen, 1983; Yunus and Khan, 2011a). For small sample sizes, the computation of the power function and the size of the test after a pre-test (PT) requires the cdf of a correlated bivariate noncentral $F$ (BNCF) distribution, which has not been reported in the literature because unlike those for the bivariate central $F$ (BCF) distribution, the formulae for the pdf and cdf of the correlated BNCF distribution are more complex; hence, there are no easy computational formulae available. As such, no statistical packages include this distribution.

Yunus and Khan (2011b) derived the bivariate noncentral chi-square (BNCC) distribution by compounding the Poisson distribution with the correlated bivariate central chi-square distribution, aiming to compute the power function of the test after pre-testing. Therefore, using the same method of derivation, we derive the pdf and cdf of the *singly* and *doubly* correlated BNCF distributions in this article. The *doubly* correlated BNCF is defined by mixing the correlated BNCC distribution with an independent central chi-square distribution. This definition allows for two noncentrality parameters from the two correlated noncentral chi-square variables in the numerator of the noncentral $F$ variables. Additionally, by compounding the BCF and Poisson distributions, we derive the *singly* correlated BNCF distribution. This form of the BNCF distribution has only one noncentrality parameter. We also propose the computational formulae of the pdf and cdf of the correlated BNCF distribution and illustrate their application in the derivation of the power function of the pre-test test (PTT) (for details on the PTT, see Khan and Pratikno, 2013). The R codes are written to compute the values of the pdf and cdf of the correlated BNCF distribution and the power curve of the PTT of the MSRM. In addition to suggesting the computational formulae, we also compute and tabulate the critical values of the distribution for selected values of the parameters and significance levels using the R codes.

The next section derives the expression for the pdf and cdf of the correlated BNCF distribution. The computational method and graphical presentation of the pdf and cdf and the critical values of the correlated BNCF distribution for different values of the noncentrality parameter are presented in Section 3. An application of the BNCF distribution to the power function of the PTT is discussed in Section 4, and concluding remarks are provided in Section 5.

## 2. The Bivariate Noncentral $F$ Distribution

In this section, the cdfs of the *singly* and *doubly* correlated bivariate noncentral $F$ distributions are obtained using the compounding of distributions technique. The *doubly* correlated BNCF is obtained by compounding the correlated BNCC distribution with an independent central chi-square distribution, thus allowing for two noncentrality parameters and a correlation coefficient parameter. Additionally, by compounding the BCF and Poisson distributions, the *singly* correlated BNCF distribution is derived. This form of the BNCF distribution has only one noncentrality parameter and a correlation coefficient parameter.

### 2.1 *The Singly Correlated Bivariate Noncentral* F *Distribution*

Let a random variable $X_i$, $i = 1, 2$ follow an $F$ distribution with $(\nu_i, \nu_2)$ degrees of freedom, and let another random variable $R$ follow a Poisson distribution with mean $\lambda$. The proposed *singly* correlated BNCF distribution is an extension of the univariate noncentral $F$ distribution introduced by Krishnaiah (1965a) and Johnson et al. (1995) for the bivariate case. The pdf of the *singly* BNCF distribution with noncentrality parameter $\lambda$ and correlation $\rho$ is defined as

$$f(x_1, x_2, \nu_r, \nu_2, \lambda) = \sum_{r=0}^{\infty} \left( \frac{e^{-\lambda/2} \left(\frac{\lambda}{2}\right)^r}{r!} \right) f_1(x_1, x_2, \nu_r, \nu_2), \tag{2.1}$$

where $f_1(x_1, x_2, \nu_r, \nu_2)$ is the pdf of a BCF distribution with $\nu_r$ and $\nu_2$ degrees of freedom in which $\nu_r = \nu_1 + 2r$, that is,

$$f_1(x_1, x_2, \nu_r, \nu_2) = \left( \frac{\nu_2^{\nu_2/2}(1-\rho^2)^{(\nu_r+\nu_2)/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)} \right) \sum_{j=0}^{\infty} \left( \frac{\rho^{2j}\Gamma(\nu_r + (\nu_2/2) + 2j)}{j!\Gamma((\nu_r/2) + j)} \right) \nu_r^{\nu_r+2j}$$
$$\times \left( \frac{(x_1x_2)^{(\nu_r/2)+j-1}}{[\nu_2(1-\rho^2) + \nu_r(x_1+x_2)]^{\nu_r+(\nu_2/2)+2j}} \right).$$

Note that the density function of the *singly* correlated BNCF distribution is obtained by compounding the BCF distribution with the Poisson probabilities.

Therefore, the cdf of the *singly* correlated BNCF distribution is defined as

$$\begin{aligned} P(.) &= P(X_1 < d, X_2 < d, \nu_r, \nu_2, \lambda) \\ &= \sum_{r=0}^{\infty} \left( \frac{e^{-\lambda/2} \left(\frac{\lambda}{2}\right)^r}{r!} \right) P_2(X_1 < d, X_2 < d, \nu_r, \nu_2), \end{aligned} \tag{2.2}$$

where

$$P_2(X_1 < d, X_2 < d, \nu_r, \nu_2) = \left( \frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)} \right) \sum_{j=0}^{\infty} \left( \frac{\rho^{2j}\Gamma(\nu_r + (\nu_2/2) + 2j)}{j!\Gamma((\nu_r/2) + j)} \right) L_{jr}$$

and $L_{jr}$ is defined as

$$L_{jr} = \int_0^{h_r} \int_0^{h_r} \frac{(x_1x_2)^{(\nu_r/2)+j-1} dx_1 dx_2}{(1 + x_1 + x_2)^{\nu_r+(\nu_2/2)+2j}}$$

with $h_r = \frac{d\nu_r}{\nu_2(1-\rho^2)}$.

For the computation of the value of the cdf of the *singly* correlated BNCF distribution, R codes are used. To make the computations easier, we represent the formula of the cdf of the *singly* correlated BNCF distribution in Eq. (2.2) as the sum of infinite series as follows:

$$
\begin{aligned}
P(.) &= \sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)\left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right)\sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_r+(\nu_2/2)+2j)}{j!\Gamma((\nu_r/2)+j)}\right)L_{jr} \\
&= \sum_{r=0}^{\infty} T_r\left[\left(\frac{1\Gamma(\nu_r+(\nu_2/2))}{0!\Gamma((\nu_r/2))}\right)L_{0r}+\left(\frac{\rho^2\Gamma(\nu_r+(\nu_2/2)+2)}{1!\Gamma((\nu_r/2)+1)}\right)L_{1r}+\cdots\cdots\right] \\
&= \sum_{r=0}^{\infty} T_r\left[H_{0r}L_{0r}+H_{1r}L_{1r}+H_{2r}L_{2r}+\ \cdots\cdots\right] \\
&= \sum_{r=0}^{\infty} T_rH_{0r}L_{0r}+T_rH_{1r}L_{1r}+T_rH_{2r}L_{2r}+\ \cdots\cdots \\
&= [T_0H_{00}L_{00}+T_0H_{10}L_{10}+T_0H_{20}L_{20}\ +\ \cdots\cdots] \\
&\quad +[T_1H_{01}L_{01}+T_1H_{11}L_{11}+T_1H_{21}L_{21}\ +\ \cdots\cdots] \\
&\quad +[T_2H_{02}L_{02}+T_2H_{12}L_{12}+T_2H_{22}L_{22}\ +\ \cdots\cdots] \\
&\quad +\cdots\cdots,
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
T_r &= \left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right)\left(\frac{(1-\rho^2)^{\nu_r/2}}{\Gamma(\nu_r/2)\Gamma(\nu_2/2)}\right), \\
H_{0r} &= \frac{1\Gamma(\nu_r+(\nu_2/2))}{0!\Gamma((\nu_r/2))},\ H_{1r}=\frac{\rho^2\Gamma(\nu_r+(\nu_2/2)+2)}{1!\Gamma((\nu_r/2)+1)}, \\
H_{2r} &= \frac{\rho^4\Gamma(\nu_r+(\nu_2/2)+4)}{2!\Gamma((\nu_r/2)+2)},\ \cdots,
\end{aligned}
$$

and $P_0$ is defined as

$$
\begin{aligned}
P_0 &= \sum_{r=0}^{\infty} T_rH_{0r}L_{0r} = T_0H_{00}L_{00}+T_1H_{01}L_{01}+T_2H_{02}L_{02}+\cdots\cdots,\ \text{ for } j=0, \\
&= \left(\frac{e^{-\lambda/2}}{0!}\times\frac{(1-\rho^2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\right)\left(\frac{1\Gamma(\nu_1+\nu_2/2)}{0!\Gamma(\nu_1/2)}\right)L_{00} \\
&\quad +\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)}{1}\times\frac{(1-\rho^2)^{\nu_1+2/2}}{\Gamma((\nu_1+2)/2)\Gamma(\nu_2/2)}\right)\left(\frac{1\Gamma(\nu_1+2+(\nu_2/2))}{0!\Gamma((\nu_1+2)/2)}\right)L_{01}+\cdots.
\end{aligned}
\tag{2.4}
$$

Similarly, we obtain the expressions for $P_1=\sum_{r=0}^{\infty}T_rH_{1r}L_{1r}$, $P_2=\sum_{r=0}^{\infty}T_rH_{2r}L_{2r}$ and so on. Finally, we write

$$
P(.) = P_0+P_1+P_2+P_3+\ \cdots\cdots = \sum_{j=0}^{\infty}P_j.
$$

Some properties of the *singly* correlated BNCF distribution are given as follows:

(i) Note that $f_1(x_1, x_2, \nu_r, \nu_2)$ in Eq. (2.1) is a pdf of a BCF distribution with $\nu_r$ and $\nu_2$ degrees of freedom; hence, $\int_0^\infty \int_0^\infty f_1(x_1, x_2, \nu_r, \nu_2)dx_1dx_2 = \lim_{d\to\infty} P_2(X_1 < d, X_2 < d)$ in Eq. (2.2) is equal to one. Furthermore, $\sum_{r=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^r}{r!} = 1$. Thus, it can be easily observed that $\int_0^\infty \int_0^\infty f(x_1, x_2, \nu_r, \nu_2, \lambda)dx_1dx_2 = 1$.

(ii) Because $f_1(x_1, x_2, \nu_r, \nu_2)$ is a pdf of a BCF, $f_1(\cdot) \geq 0$. It is noted that the quantity $\frac{e^{-\lambda/2}(\lambda/2)^r}{r!}$ is always positive. Therefore, $f(\cdot) \geq 0$.

(iii) From Eq. (2.1), the central case of the bivariate $F$ distribution proposed by Krishnaiah (1965a) is a special case of the *singly* correlated BNCF distribution when the noncentrality parameter, $\lambda$, is equal to zero.

## 2.2 *The Doubly Correlated Bivariate Noncentral* F *Distribution*

Let the random variables $(X_1, X_2)$ jointly follow a correlated BNCC distribution with $m$ degrees of freedom, noncentrality parameters $\theta_1$ and $\theta_2$, and a correlation coefficient $\rho$, and let the random variable $Z$ follow a central chi-square distribution with $n$ degrees of freedom. We propose the cdf of the *doubly* correlated BNCF by compounding the two aforementioned distributions.

The pdf of the correlated BNCC variables $X_1$ and $X_2$ proposed by Yunus and Khan (2011b) is given by

$$g(x_1, x_2) = \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\left[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)\right] \times \left[\frac{(x_1)^{m/2+j+r_1-1}e^{-\frac{(x_1)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right] \times \left[\frac{(x_2)^{m/2+j+r_2-1}e^{-\frac{(x_2)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right], \qquad (2.5)$$

and the pdf of a central chi-square variable $Z$ with $n$ degrees of freedom is given by

$$f(z) = \frac{z^{(n/2)-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)}, \qquad (2.6)$$

where $Z$ is independent of $X_1$ and $X_2$.

Therefore, the random variables $(Y_1, Y_2)$, where

$$Y_i = \frac{X_i/m}{Z/n}, \quad \text{for } i = 1, 2, \qquad (2.7)$$

have a joint cdf given by

$$P(Y_1 \leq a, Y_2 \leq b) = \int_{z=0}^{\infty} f(z) \int_{x_2=0}^{\frac{bmz}{n}} \int_{x_1=0}^{\frac{amz}{n}} g(x_1, x_2)dx_1dx_2dz. \qquad (2.8)$$

The distribution function given in Eq. (2.8) is the cdf of the proposed *doubly* correlated BNCF distribution with $m$ and $n$ degrees of freedom, noncentrality parameters $\theta_1$ and $\theta_2$, and correlation coefficient $\rho$.

In addition, Eq. (2.8) can be expressed as the following sum of infinite series

$$P(Y_1 \le a, Y_2 \le b) = (1-\rho^2)^{\frac{m}{2}} \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!}\rho^{2j} I_2(\tilde{\alpha}_j, \tilde{c}, \beta) \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}, \tag{2.9}$$

where

$$I_2(\tilde{\alpha_j}, \tilde{c}, \beta) = \int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\gamma(\alpha_1, c_1 z)}{\Gamma(\alpha_1)}\frac{\gamma(\alpha_2, c_2 z)}{\Gamma(\alpha_2)}dz$$

and

$$\beta = \frac{n}{2}, \quad \tilde{c} = \left(\frac{am}{n(1-\rho^2)}, \frac{bm}{n(1-\rho^2)}\right), \quad \tilde{\alpha}_j = \left(\frac{m}{2}+j+r_1, \frac{m}{2}+j+r_2\right).$$

Here, $\gamma(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1}dt$ and $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$.

To ease the computational difficulties of the cdf, we use the following form of $I_2$ given by Amos and Bulgren (1972),

$$I_2(\tilde{\alpha}_j, \tilde{c}, \beta) = I_u(\alpha_1, \beta) - \frac{(1-u)^\beta}{\alpha_1}\frac{\Gamma(\beta+\alpha_1)}{\Gamma(\beta)\Gamma(\alpha_1)} \times \sum_{r=0}^{\infty}\frac{(\beta+\alpha_1)_r}{(1+\alpha_1)_r}u^{r+\alpha_1} I_{1-y}(r+\beta+\alpha_1, \alpha_2), \tag{2.10}$$

with

$$u = c_1/(1+c_1), \quad 1-y = (1+c_1)/(1+c_1+c_2),$$

and

$$\int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\gamma(\alpha, cz)}{\Gamma(\alpha)}dz = I_z(\alpha, \beta) \quad \text{and}$$

$$\int_0^\infty \frac{e^{-z}z^{\beta-1}}{\Gamma(\beta)}\frac{\Gamma(\alpha, cz)}{\Gamma(\alpha)}dz = I_{1-z}(\beta, \alpha)$$

are the regularized beta functions, with $\alpha > 0$, $\beta > 0$, $x = c/(1+c)$, and $1-x = 1/(1+c)$.

See the Appendix for the pdf of the *doubly* correlated BNCF distribution, which is derived using the transformation of variables method.

Some properties of the *doubly* correlated BNCF distribution are given as follows:

(i) From Eq. (2.9), we find that $F_1(a, b; r_1, r_2) = (1-\rho^2)^{\frac{m}{2}}\sum_{j=0}^{\infty}\frac{(\frac{m}{2})_j}{j!}\rho^{2j} I_2(\tilde{\alpha}_j, \tilde{c}, \beta)$ is the cdf of a BCF distribution (Amos and Bulgren, 1972); thus, $F_1(a, b)$ approaches 1 as both $a$ and $b$ go to infinity. It is clear that both quantities $\sum_{r_1=0}^{\infty}\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}$ and $\sum_{r_2=0}^{\infty}\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}$ are equal to one. It follows that the cdf of the *doubly* correlated BNCF distribution approaches 1 as both $a$ and $b$ go to infinity.

(ii) When $a$ and $b$ are zero, it is easy to show that the cdf of the *doubly* correlated BNCF is zero.

(iii) Note that $F_1(a, b)$ is an increasing function because it is a cdf of a BCF distribution. It follows that the cdf of the *doubly* correlated BNCF distribution is also an increasing function.

(iv) As $b$ approaches infinity, $\gamma(\alpha_2, c_2 z) = \Gamma(\alpha_2)$ through $c_2 = bm/n(1-\rho^2)$, and the second term on the right-hand side of Eq. (2.10) becomes zero because $1 - y$ approaches zero as $b$ goes to infinity. Further simplifications yield the following marginal distribution function:

$$\sum_{r_1=0}^{\infty} I_{\frac{my_1}{n+my_1}}(m/2 + r_1, n/2) \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} = F(y_1; m, n, \theta_1), \tag{2.11}$$

which is the cdf of the noncentral $F$ distribution of $Y_1$, with noncentrality parameter $\theta_1$ and degrees of freedom $m$ and $n$. In the same manner, the marginal distribution function for $Y_2$ can be derived.

(v) The central $F$ distribution can be obtained from the noncentral distribution if the noncentrality parameters, $\theta_1$ and $\theta_2$, are equal to 0. Because $r_1$ and $r_2$ are both zero, we rewrite (2.9) as

$$P(Y_1 \le a, Y_2 \le b) = (1-\rho^2)^{\frac{m}{2}} \sum_{j=0}^{\infty} \frac{(\frac{m}{2})_j}{j!} \rho^{2j} I_2(\hat{\alpha}_j, \hat{c}, \beta)$$

with

$$\beta = \frac{n}{2}, \quad \hat{c} = \left( \frac{am}{(1-\rho^2)n}, \frac{bm}{(1-\rho^2)n} \right), \quad \hat{\alpha} = \left( \frac{m}{2} + j, \frac{m}{2} + j \right). \tag{2.12}$$

Thus, we arrive at the central correlated bivariate $F$ distribution proposed by Amos and Bulgren (1972) after allowing both noncentrality parameters equal zero in the *doubly* correlated BNCF distribution.

(vi) For $\rho = 0$, which implies $j = 0$, we write Eq. (2.9) as

$$P(Y_1 \le a, Y_2 \le b) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} I_2(\check{\alpha}, \check{c}, \beta) \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!} \tag{2.13}$$

with

$$\beta = \frac{n}{2}, \quad \check{c} = \left( \frac{am}{n}, \frac{bm}{n} \right), \quad \check{\alpha} = \left( \frac{m}{2} + r_1, \frac{m}{2} + r_2 \right). \tag{2.14}$$

It can be observed that $Y_1$ and $Y_2$ are not independent, although the correlation coefficient between $Y_1$ and $Y_2$ is zero. In other words, the *doubly* correlated BNCF can have a zero correlation, but the marginal distributions do not support statistical independence.

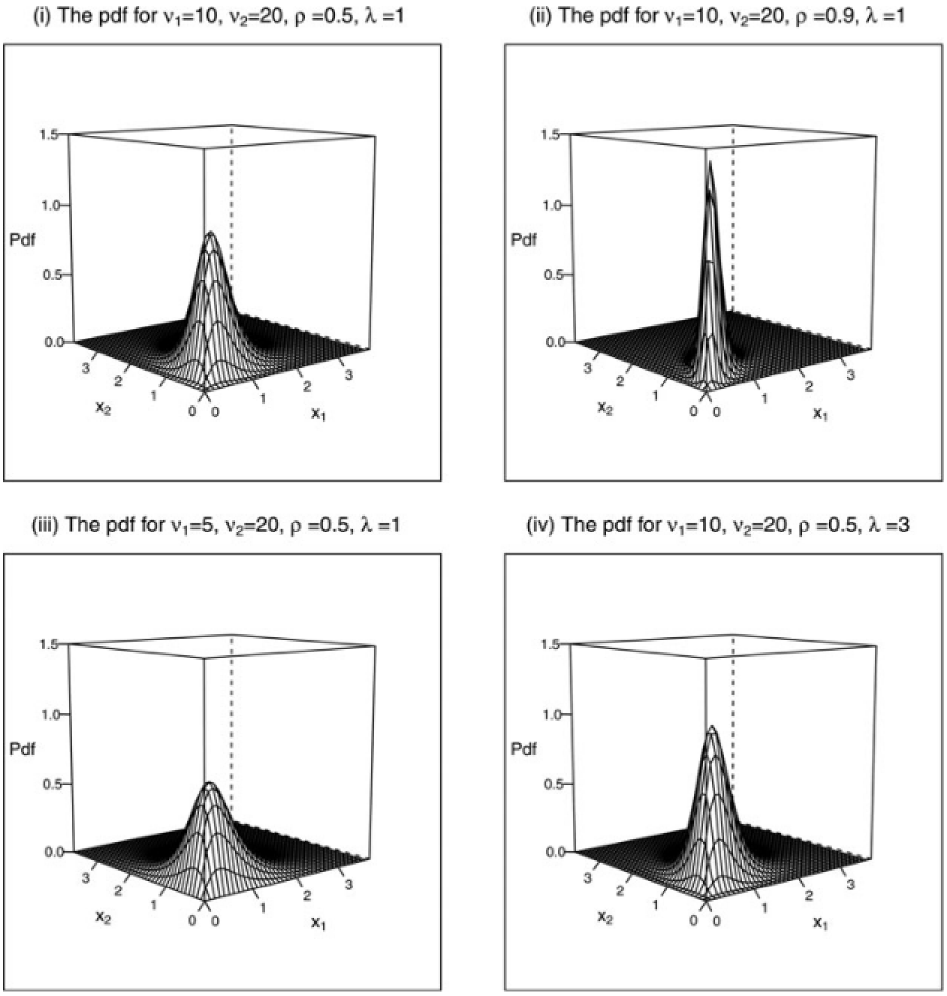


**Figure 1.** The pdf of the *singly* correlated bivariate noncentral $F$ distribution.



## 3. Computation of the PDF and CDF

To compute the values of the pdf and cdf of the BNCF distributions, R codes are written. The R package is also used for the graphical representation of the pdf and cdf. The pdf of the *singly* BNCF distribution is computed using Eq. (2.1) and plotted in Fig. 1. The graph in Fig. 1(iii) has a wider spread than that in Fig. 1(i) due to the smaller value of $\nu_1$. Comparing Fig. 1(i) and 1(iv), the spread of the distribution in Fig. 1(iv) decreases due to the increase in the value of the noncentrality parameter. As the value of $\rho$ increases, the spread of the distribution decreases and the pdf shrinks, as shown in Fig. 1(ii). For the *doubly* correlated BNCF distribution, the pdf is calculated using Eq. (A.4) and plotted in Fig. 2. The graphs in Fig. 2 show properties similar to those shown in Fig. 1 but with varying probabilities.

To compute the cdf of the *singly* correlated BNCF distribution in Eq. (2.3), we choose arbitrary values of the degrees of freedom ($\nu_1$, $\nu_2$), noncentrality parameter ($\lambda$), correlation coefficient ($\rho$), and upper limit ($d$) of the variable. Figure 3 shows that the cdf of the *singly*

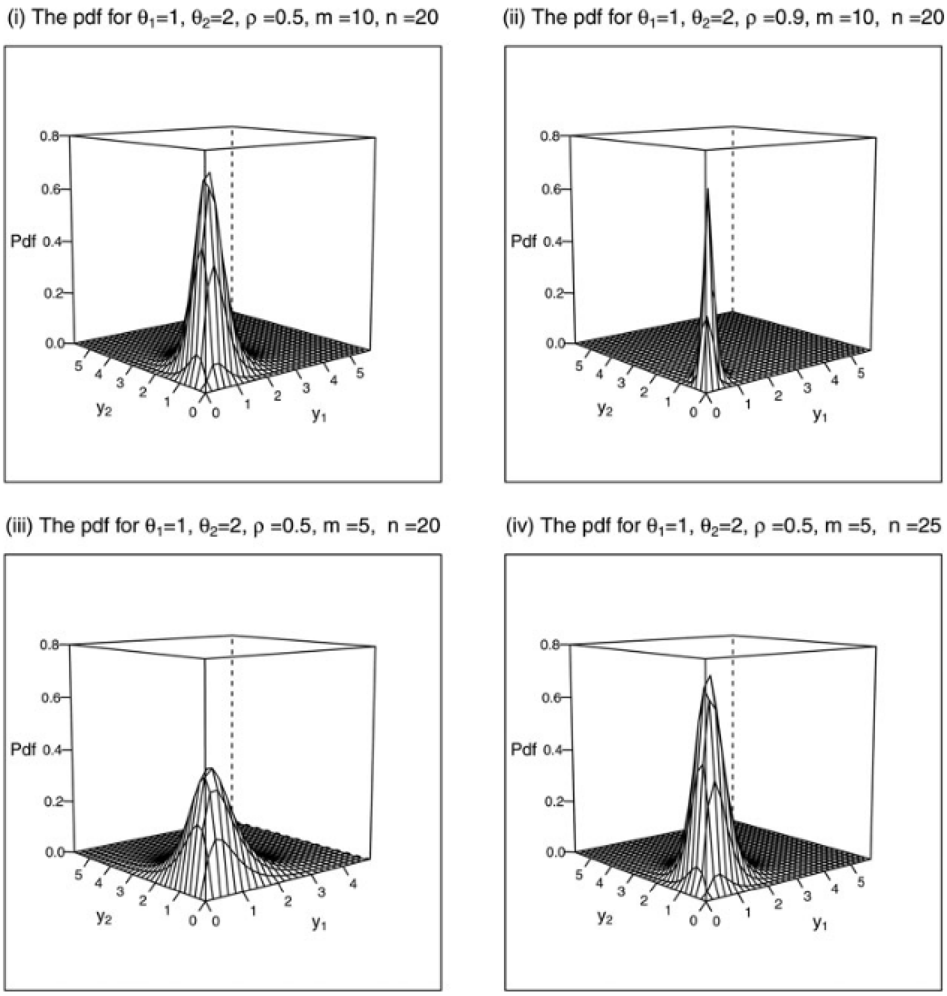


**Figure 2.** The pdf of the *doubly* correlated bivariate noncentral $F$ distribution.



correlated BNCF distribution increases as the value of any of the parameters, namely, the degrees of freedom $\nu_1$ (for fixed $\nu_2$), λ, or $d$, increases.

The cdf of the *doubly* correlated BNCF distribution is computed using Eq. (2.9) for arbitrary degrees of freedom $(m, n)$, noncentrality parameters $(\theta_1, \theta_2)$, correlation coefficient $(\rho)$, and upper limit $(a = b = d)$. The graphs of the cdf of the *doubly* correlated BNCF distribution are presented in Fig. 4. Interestingly, the cdf curve approaches 1 more rapidly for a larger correlation coefficient (see Fig. 4(i)), a smaller noncentrality parameter (see Fig. 4(ii)), and a greater number of degrees of freedom $(m, n)$ (see Fig. 4(iii) and 4(iv)). Figure 4 shows that the shape of the cdf curve is sigmoidal, which depends on the values of the noncentrality parameters $(\theta_1, \theta_2)$, degrees of freedom $(m, n)$, and correlation coefficient $(\rho)$. Tablexs 1 and 2 provide the values of $d$ for different values of $m$, $n$, $\theta_1$, $\theta_2$, and $\alpha$, where

$$P(Y_1 < d, Y_2 < d) = \int_0^d \int_0^d f(y_1, y_2) dy_1 dy_2 = (1 - \alpha),$$

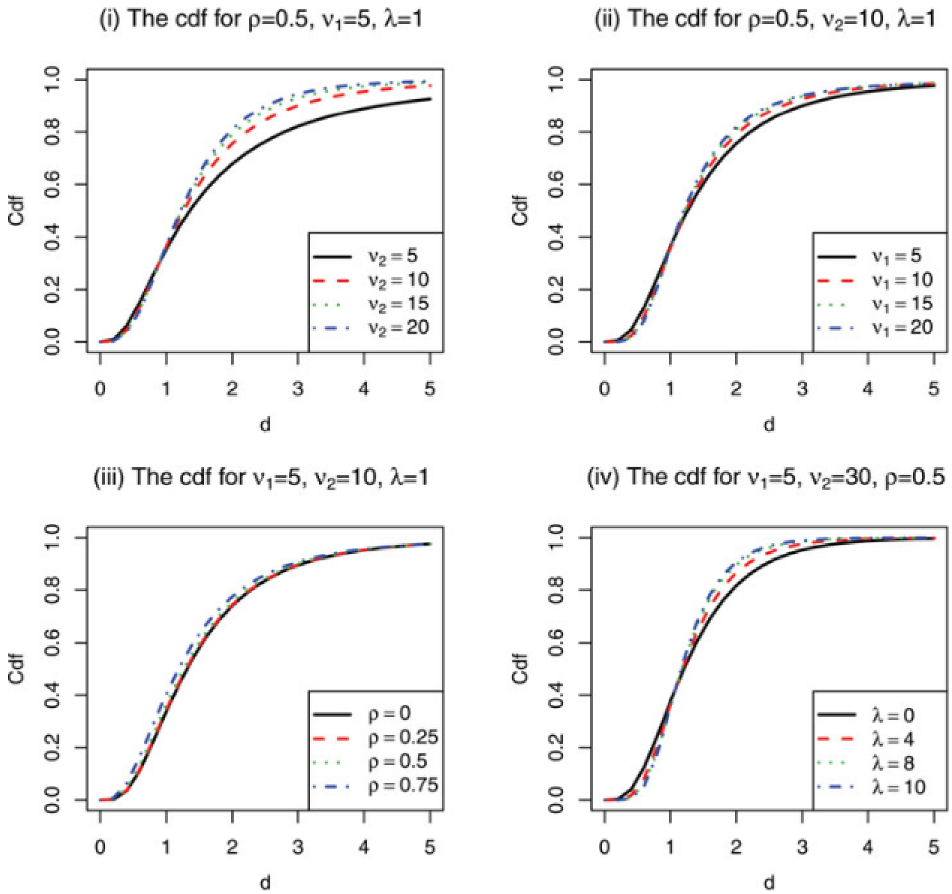


**Figure 3.** The cdf of the singly correlated BNCF distribution with arbitrary values of $\rho$, $\lambda$, $\nu_1$, and $\nu_2$.

for the case in which $\rho = 0.5$.

## 4. Application to the Power Function of the PTT

To test the null hypothesis on the intercept vector $H_0 : \boldsymbol{\beta}_0 = \boldsymbol{\beta}_{00}$ (given known vector) against $H_a : \boldsymbol{\beta}_0 > \boldsymbol{\beta}_{00}$ in the multivariate simple regression model

$$\boldsymbol{y}_i = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x_i + \boldsymbol{e}_i, \tag{4.1}$$

(for details, see Khan, 2006) when there is nonsample prior information on the slope vector $\boldsymbol{\beta}_1$, the test statistic follows a correlated bivariate $F$ distribution. The ultimate test on $H_0$ is called the pre-test test (PTT) because it depends on the outcome of the pre-test on the suspected slope, that is, $H_0^* = \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10}$ (see Khan and Pratikno, 2013). The cdf of the *doubly* correlated BNCF distribution is involved in the formula for the power function of the PTT. To illustrate the method, we conduct a simulation study by generating random data using the R package.

The explanatory variable ($x$) is generated from the uniform distribution between 0 and 1. The error vector ($\boldsymbol{e}$) is generated from a $p = 3$ dimensional multivariate normal

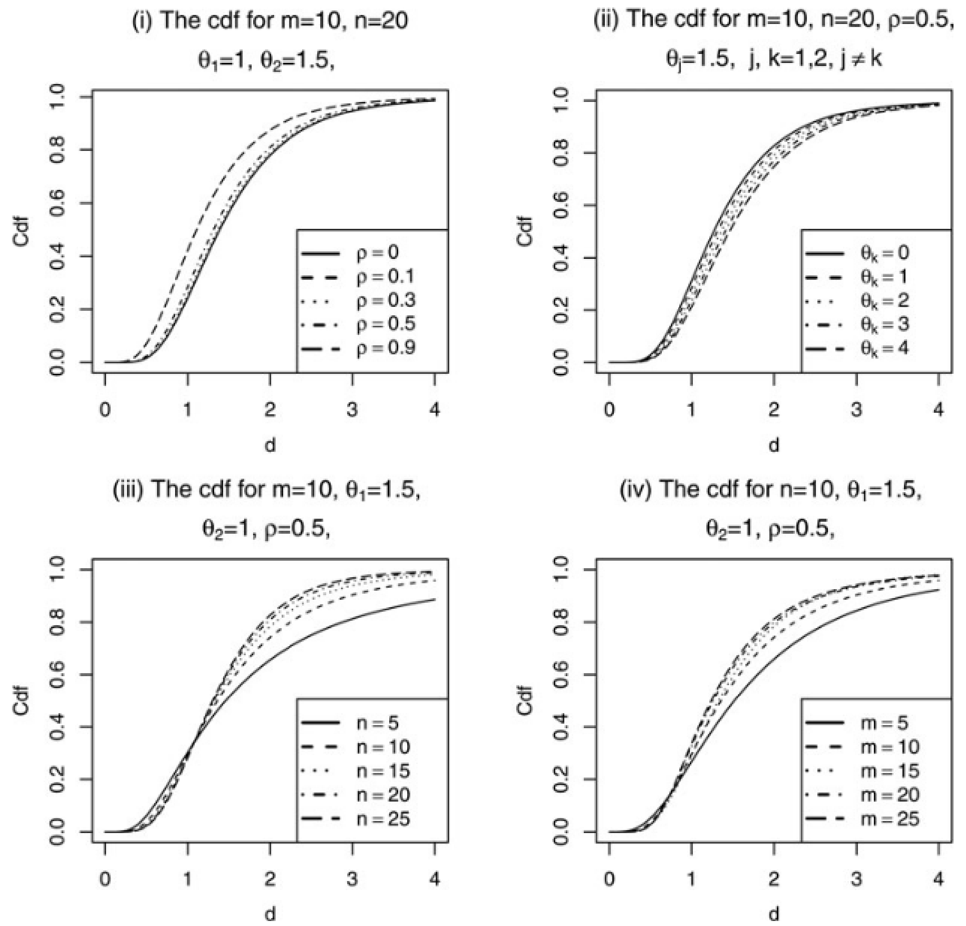


**Figure 4.** The cdf of the doubly correlated BNCF distribution with arbitrary values of $\rho$, $\theta_k$, $m$, and $n$.

distribution with $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_3$, where $I_3$ is the identity matrix of order 3. Then, the dependent variable $(y_1)$ is determined by $y_1 = \beta_0^{'} + \beta_1^{'} x + e_1$ for $\beta_0^{'} = 3$ and $\beta_1^{'} = 1.5$. Similarly, $y_2$ and $y_3$ are determined by $y_2 = \beta_0^{''} + \beta_1^{''} x + e_2$ for $\beta_0^{''} = 5$ and $\beta_1^{''} = 2.5$ and by $y_3 = \beta_0^{'''} + \beta_1^{'''} x + e_3$ for $\beta_0^{'''} = 6$ and $\beta_1^{'''} = 3$. For each of the three cases, $n = 20$ random variates are generated.

Considering the three cases (i) unspecified $\boldsymbol{\beta}_1$, (ii) specified $\boldsymbol{\beta}_1$, and (iii) uncertain prior information on $\boldsymbol{\beta}_1$, we define the unrestricted, restricted, and pre-test test statistics as follows: $T^{\mathrm{UT}} = \sum_{i=1}^{n}(x_i - \bar{x})^2[(\tilde{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_{00})'\hat{\Sigma}^{-1}(\tilde{\boldsymbol{\beta}}_0 - \boldsymbol{\beta}_{00})], T^{\mathrm{RT}} = \sum_{i=1}^{n}(x_i - \bar{x})^2[(\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})'\hat{\Sigma}^{-1}(\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})]$, and $T^{\mathrm{PT}} = \sum_{i=1}^{n}(x_i - \bar{x})^2[(\tilde{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10})'\hat{\Sigma}^{-1}(\tilde{\boldsymbol{\beta}}_1 - \boldsymbol{\beta}_{10})]$, respectively. Here, $\hat{\Sigma}^{-1} = \frac{1}{n-p}\sum_{i=1}^{n}(\boldsymbol{y}_i - \tilde{\boldsymbol{\beta}}_0 - \tilde{\boldsymbol{\beta}}_1 x_i)(\boldsymbol{y}_i - \tilde{\boldsymbol{\beta}}_0 - \tilde{\boldsymbol{\beta}}_1 x_i)'$, where $\hat{\boldsymbol{\beta}}_0 = \bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x}$, $\tilde{\boldsymbol{\beta}}_0 = \bar{\boldsymbol{y}} - \tilde{\boldsymbol{\beta}}_1 \bar{x}$, $\tilde{\boldsymbol{\beta}}_1 = \sum_{i=1}^{n} \frac{(\boldsymbol{y}_i - \bar{\boldsymbol{y}})(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$, $\bar{x} = \sum_{i=1}^{n} x_i/n$, and $\bar{\boldsymbol{y}} = \sum_{i=1}^{n} \boldsymbol{y}_i/n$.

Under $H_a : \boldsymbol{\beta}_0 > \boldsymbol{\beta}_{00}$, $T^{\mathrm{UT}}$ and $T^{\mathrm{RT}}$ follow a noncentral $F$ distribution with $(p, n-p)$ degrees of freedom and noncentrality parameters $\Delta_1^2/2$ and $\Delta_2^2/2$, respectively, where $\Delta_1^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2[(\boldsymbol{\beta}_0 - \boldsymbol{\beta}_{00})'\Sigma^{-1}(\boldsymbol{\beta}_0 - \boldsymbol{\beta}_{00})]$ and $\Delta_2^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2[(\bar{Y} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})'\hat{\Sigma}^{-1}(\bar{\boldsymbol{y}} - \boldsymbol{\beta}_{10}\bar{x} - \boldsymbol{\beta}_{00})]$. Under $H_a^{\star} : \boldsymbol{\beta}_1 > \boldsymbol{\beta}_{10}$, $T^{\mathrm{PT}}$ follows a noncentral $F$ distribution with $(p, n-p)$

**Table 1**
Percentage points for the *doubly* correlated BNCF distribution for $\rho = 0.5$ and $\alpha = 0.05$

| | | $\theta_1$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | | | 4 | | | 10 | | |
| | | $\theta_2$ | | | $\theta_2$ | | | $\theta_2$ | | |
| $m$ | $n$ | 2 | 4 | 10 | 2 | 4 | 10 | 2 | 4 | 10 |
| 2 | 5 | 11.9 | 14.1 | 22.8 | 14.5 | 16.2 | 23.4 | 23.6 | 24.2 | 27.9 |
| | 6 | 10.4 | 12.3 | 19.8 | 12.6 | 14.0 | 20.3 | 20.4 | 20.8 | 24.0 |
| | 8 | 8.8 | 10.4 | 16.6 | 10.6 | 11.8 | 17.0 | 16.9 | 17.3 | 19.8 |
| | 10 | 8.4 | 9.5 | 15.0 | 9.6 | 10.6 | 15.2 | 15.1 | 15.4 | 17.6 |
| 4 | 5 | 8.4 | 9.5 | 13.6 | 9.7 | 10.5 | 14.0 | 14.1 | 14.4 | 16.3 |
| | 6 | 7.3 | 8.2 | 11.1 | 8.3 | 9.0 | 12.0 | 12.1 | 12.3 | 14.0 |
| | 8 | 6.1 | 6.8 | 9.8 | 6.9 | 7.5 | 10.0 | 10.0 | 10.2 | 11.4 |
| | 10 | 5.4 | 6.1 | 8.8 | 6.2 | 6.7 | 8.9 | 8.9 | 9.0 | 10.1 |
| 6 | 5 | 7.2 | 7.9 | 10.6 | 8.0 | 8.6 | 10.9 | 10.9 | 11.1 | 12.4 |
| | 6 | 6.2 | 6.8 | 9.1 | 6.9 | 7.3 | 9.3 | 9.3 | 9.5 | 10.6 |
| | 8 | 5.1 | 5.6 | 7.5 | 5.7 | 6.0 | 7.7 | 7.6 | 7.8 | 8.7 |
| | 10 | 4.6 | 5.0 | 6.7 | 5.1 | 5.4 | 6.8 | 6.8 | 6.9 | 7.7 |
| 8 | 5 | 6.6 | 7.1 | 9.0 | 7.2 | 7.6 | 9.3 | 9.3 | 9.5 | 10.5 |
| | 6 | 5.6 | 6.1 | 7.7 | 6.2 | 6.5 | 8.0 | 8.0 | 8.1 | 8.9 |
| | 8 | 4.7 | 5.0 | 6.4 | 5.1 | 5.3 | 6.5 | 6.5 | 6.6 | 7.3 |
| | 10 | 4.1 | 4.5 | 5.7 | 4.5 | 4.7 | 5.8 | 5.8 | 5.9 | 6.4 |
| 10 | 5 | 6.2 | 6.7 | 8.1 | 6.7 | 7.0 | 8.3 | 8.4 | 8.5 | 9.3 |
| | 6 | 5.3 | 5.7 | 6.9 | 5.7 | 6.0 | 7.1 | 7.1 | 7.3 | 7.9 |
| | 8 | 4.4 | 4.6 | 5.7 | 4.7 | 4.9 | 5.8 | 5.8 | 5.9 | 6.4 |
| | 10 | 3.9 | 4.1 | 5.1 | 4.1 | 4.3 | 5.2 | 5.1 | 5.2 | 5.7 |

degrees of freedom and noncentrality parameter $\Delta_3^2/2$, where $\Delta_3^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2[(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_{10})'\hat{\Sigma}^{-1}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_{10})]$.

Let $\{K_n\}$ be a sequence of alternative hypotheses

$$K_n : \boldsymbol{\beta}_0 = \boldsymbol{\beta}_{00} + \boldsymbol{\lambda}_1/\sqrt{n}, \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10} + \boldsymbol{\lambda}_2/\sqrt{n}, \tag{4.2}$$

where $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are vectors of fixed real numbers. Under $\{K_n\}$, the power function of the PTT is given by

$$\begin{aligned}\pi^{\text{PTT}}(\boldsymbol{\lambda}) &= P\left(T^{\text{PT}} < a, T^{\text{RT}} > c\right) + P\left(T^{\text{PT}} \geq a, T^{\text{UT}} > b\right)\\ &= P\left(T^{\text{PT}} < a\right) \; P\left(T^{\text{RT}} > c\right) + d_{1r}\,(a, b, \rho)\\ &= \left[1 - P\left(T^{\text{PT}} > a\right)\right] \; P\left(T^{\text{RT}} > c\right) + d_{1r}\,(a, b, \rho)\,,\end{aligned} \tag{4.3}$$

where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1/\sqrt{n}, \boldsymbol{\lambda}_2/\sqrt{n})$, $a = F_{\alpha_3,p,n-p} - \phi_2$, $b = F_{\alpha_1,p,n-p} - \phi_1$, and $c = F_{\alpha_2,n,n-1} - [\phi_1 + \phi_2\bar{x}] + \omega\bar{x}$ for $\phi_1 = \boldsymbol{\lambda}_1'\Sigma^{-1}\boldsymbol{\lambda}_1$, $\phi_2 = \boldsymbol{\lambda}_2'\Sigma^{-1}\boldsymbol{\lambda}_2$, $\omega = \boldsymbol{\lambda}_1'\Sigma^{-1}\boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_2'\Sigma^{-1}\boldsymbol{\lambda}_1$, and

**Table 2**
Percentage points for the *doubly* correlated BNCF distribution for $\rho = 0.5$ and $\alpha = 0.01$

| | | $\theta_1$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | | | 4 | | | 10 | | |
| | | $\theta_2$ | | | $\theta_2$ | | | $\theta_2$ | | |
| $m$ | $n$ | 2 | 4 | 10 | 2 | 4 | 10 | 2 | 4 | 10 |
| 2 | 5 | 26.0 | 30.7 | 49.1 | 31.4 | 35.0 | 50.3 | 50.5 | 51.6 | 59.3 |
| | 6 | 20.9 | 24.7 | 39.0 | 25.1 | 27.7 | 39.8 | 39.8 | 40.6 | 46.5 |
| | 8 | 15.9 | 18.8 | 29.3 | 19.0 | 20.9 | 29.8 | 29.7 | 30.2 | 34.3 |
| | 10 | 13.6 | 16.0 | 24.7 | 16.1 | 17.6 | 25.1 | 24.9 | 25.3 | 28.6 |
| 4 | 5 | 18.0 | 20.3 | 29.0 | 20.6 | 22.3 | 29.7 | 29.8 | 30.4 | 34.4 |
| | 6 | 14.3 | 16.1 | 22.9 | 16.3 | 17.6 | 23.4 | 23.4 | 23.9 | 26.8 |
| | 8 | 10.7 | 12.0 | 17.0 | 12.2 | 13.1 | 17.4 | 17.3 | 17.6 | 19.7 |
| | 10 | 9.0 | 10.1 | 14.3 | 10.2 | 11.0 | 14.5 | 14.4 | 14.6 | 16.3 |
| 6 | 5 | 15.3 | 16.8 | 22.4 | 17.0 | 18.1 | 23.0 | 23.0 | 23.5 | 26.1 |
| | 6 | 12.0 | 13.2 | 17.6 | 13.3 | 14.2 | 18.0 | 17.9 | 18.3 | 20.3 |
| | 8 | 8.9 | 9.7 | 13.0 | 9.9 | 10.5 | 13.2 | 13.2 | 13.5 | 14.9 |
| | 10 | 7.5 | 8.2 | 10.8 | 8.2 | 8.7 | 11.0 | 10.9 | 11.1 | 12.3 |
| 8 | 5 | 14.0 | 15.0 | 19.1 | 15.2 | 16.0 | 19.6 | 19.6 | 20.0 | 21.9 |
| | 6 | 10.9 | 11.7 | 14.9 | 11.8 | 12.5 | 15.2 | 15.2 | 15.5 | 17.1 |
| | 8 | 8.1 | 8.7 | 11.0 | 8.7 | 9.2 | 11.2 | 11.1 | 11.3 | 12.4 |
| | 10 | 6.7 | 7.2 | 9.1 | 7.2 | 7.6 | 9.3 | 9.2 | 9.4 | 10.2 |
| 10 | 5 | 13.1 | 14.0 | 17.1 | 14.1 | 14.8 | 17.5 | 17.5 | 17.8 | 19.5 |
| | 6 | 10.2 | 10.9 | 13.3 | 11.0 | 11.5 | 13.6 | 13.6 | 13.8 | 15.1 |
| | 8 | 7.5 | 8.0 | 9.8 | 8.0 | 8.4 | 10.0 | 9.9 | 10.1 | 11.0 |
| | 10 | 6.2 | 6.6 | 8.1 | 6.6 | 6.9 | 8.2 | 8.2 | 8.3 | 9.0 |



$d_{1r}(a, b, \rho)$ is a correlated bivariate $F$ probability integral defined as

$$d_{1r}(a, b, \rho) = \int_b^\infty \int_a^\infty f(F^{\text{PT}}, F^{\text{UT}}) dF^{\text{PT}} dF^{\text{UT}} \tag{4.4}$$

with $\rho = \frac{n^2(n-p-4)}{(2n-p-2)^2(n-p-4)}$. Clearly, the power of the PTT is defined in terms of the powers of the RT and PT as well as the cdf of the *doubly* correlated BNCF distribution.

Figure 5 shows the graphs of the power function of the PTT in terms of $d_{1r}(d, d, \rho)$ for selected values of the correlation coefficients ($\rho$), noncentrality parameters ($\theta_1, \theta_2$), and degrees of freedom ($m, n$). The power of the PTT decreases as the values of $\rho$ increases. The power of the PTT is identical for a fixed value of $\rho$, regardless of its sign. This figure shows that the power of the PTT increases as the values of the noncentrality parameters increase. The power of the PTT decreases as the value of the first degrees of freedom ($m$) increases and that of the second degrees of freedom ($n$) decreases.

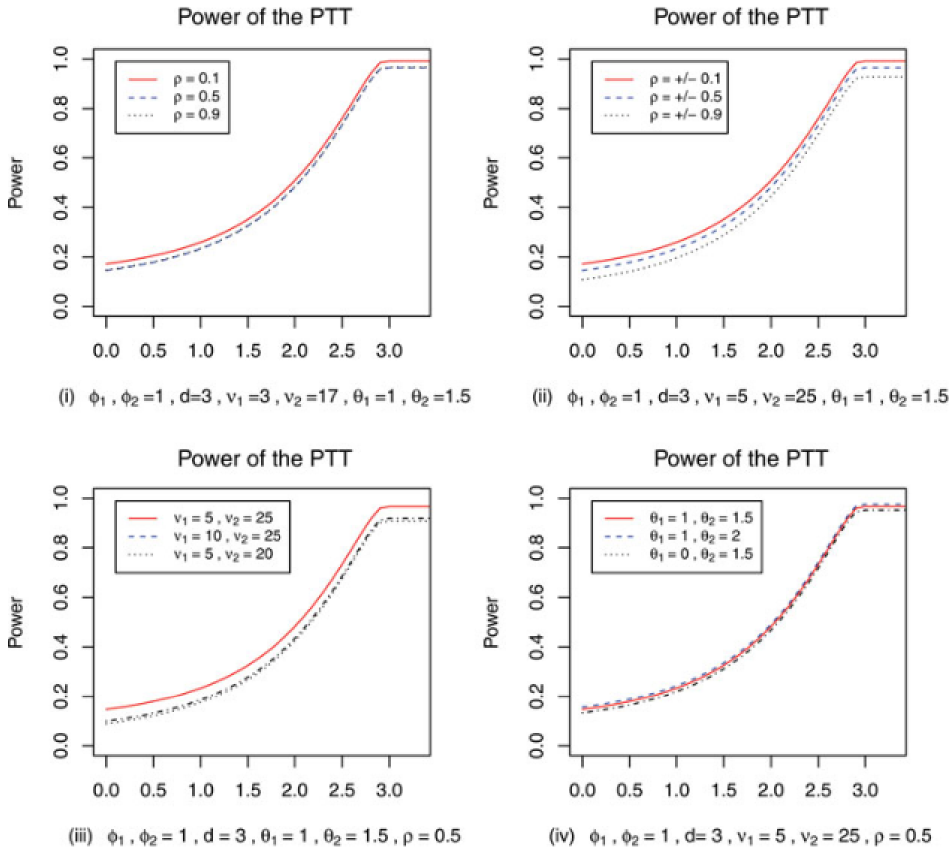


**Figure 5.** The power of the PTT using the cdf of the *doubly* correlated bivariate noncentral $F$ distribution.

## 5. Concluding Remarks

This article derives the pdf and cdf of both the *singly* and *doubly* correlated BNCF distributions. The R codes are written to calculate and plot the pdf and cdf of the distributions as well as the power function of the PTT. Two tables of critical values of the *doubly* correlated BNCF distribution for selected values of the noncentrality parameters and $\rho = 0.5$ at the significance levels 0.01 and 0.05 are presented. As an application of the distribution, the power function of the PTT for the MSRM is calculated and plotted.

The cdf of both the *singly* and *doubly* correlated BNCF distributions depend on the values of the noncentrality parameters, degrees of freedom, and correlation coefficient. The cdf curves for both *singly* and *doubly* correlated BNCF distributions are closer to one when there is an increase in the value of the degrees of freedom, correlation coefficient, and the variables for which the cdf is required. However, a smaller value of the noncentrality parameter leads to a larger value of the cdf for the *doubly* correlated BNCF distribution.

The power function of the PTT depends on the number of degrees of freedom, the correlation coefficient, and the noncentrality parameters. It decreases as the value of the correlation coefficient $\rho$, the number of degrees of freedom of the numerator $\nu_1$, or both increase, but it increases as the value of the noncentrality parameter increases.

We find that the central bivariate $F$ distribution proposed by Krishnaiah (1965a) is a special case of the proposed *singly* correlated BNCF distribution, whereas the central bivariate $F$ distribution introduced by Amos and Bulgren (1972) is a special case of the proposed *doubly* correlated BNCF distribution when the noncentrality parameters are zero. We also observe that the two variables of the BNCF distributions are not independent even if the value of $\rho$ is 0. This is another example of a case in which zero correlation between two random variables does not imply the variables' independence.

## Appendix: Derivation of the PDF of the Doubly Correlated BNCF Distribution

Using the transformation of variables method for the multivariable case (see, for instance, Wackerly et al., 2008, p. 325), we obtain the joint pdf of $\boldsymbol{y} = [y_1, y_2]'$ and $z$ variables as

$$f(\boldsymbol{y}, z) = f(\boldsymbol{x}) f(z) \mid J((\boldsymbol{x}, z) \to (\boldsymbol{y}, z)) \mid, \tag{A.1}$$

where $y_1 = \frac{nx_1}{mz}$, $y_2 = \frac{nx_2}{mz}$ and the Jacobian of the transformation $(x_1, x_2, z) \to (y_1, y_2, z)$ is given by

$$det. \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial z} \\ \frac{\partial z}{\partial y_1} & \frac{\partial z}{\partial y_2} & \frac{\partial z}{\partial z} \end{pmatrix} = \text{det.} \begin{pmatrix} \frac{m}{n}z & 0 & \frac{m}{n}y_1 \\ 0 & \frac{m}{n}z & \frac{m}{n}y_2 \\ 0 & 0 & 1 \end{pmatrix} = \left(\frac{m}{n}z\right)^2.$$

Therefore, the joint pdf of $\boldsymbol{y}$ and $z$ is given by

$$\begin{aligned} f(\boldsymbol{y}, z) = {} & \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \left[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)\right] \\ & \times \left[\frac{(\frac{m}{n}y_1 z)^{m/2+j+r_1-1} e^{-\frac{(\frac{m}{n}y_1 z)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right] \\ & \times \left[\frac{(\frac{m}{n}y_2 z)^{m/2+j+r_2-1} e^{-\frac{(\frac{m}{n}y_2 z)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right] \\ & \times \frac{z^{(n/2)-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)} \times \left(\frac{m}{n}z\right)^2. \end{aligned} \tag{A.2}$$

Thus, the density function of $\boldsymbol{y}$ is obtained as

$$f(\boldsymbol{y}) = f(y_1, y_2) = \int_z f(\boldsymbol{y}, z)dz. \tag{A.3}$$

Therefore, by applying some algebra and calculus, the pdf of the *doubly* correlated BNCF distribution becomes

$$f(y_1, y_2) = \left(\frac{m}{n}\right)^m \left[\frac{(1-\rho^2)^{\frac{m+n}{2}}}{\Gamma(m/2)\Gamma(n/2)}\right] \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \left[\frac{\rho^{2j}}{j!}\left(\frac{m}{n}\right)^{2j}\Gamma(m/2+j)\right]$$

$$\times \left[ \left( \frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!} \right) \left( \frac{\left(\frac{m}{n}\right)^{r_1}}{\Gamma(m/2+j+r_1)} \right) \left( y_1^{m/2+j+r_1-1} \right) \right]$$

$$\times \left[ \left( \frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!} \right) \left( \frac{\left(\frac{m}{n}\right)^{r_2}}{\Gamma(m/2+j+r_2)} \right) \left( y_2^{m/2+j+r_2-1} \right) \right]$$

$$\times\ \Gamma(q_{rj}) \left[ (1-\rho^2) + \frac{m}{n} y_1 + \frac{m}{n} y_2 \right]^{-(q_{rj})}, \qquad \text{(A.4)}$$

where $q_{rj} = m + (n/2) + 2j + r_1 + r_2$.

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