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by Idha Sihwaningrum

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Extension of Hardy–Littlewood–Sobolev Inequalities for Riesz Potentials on Hypergroups

Idha Sihwaningrum, Sri Maryani and Hendra Gunawan

Abstract. We establish in this paper the Hardy–Littlewood–Sobolev inequalities for the Riesz potentials on Morrey spaces over commutative hypergroups. As a consequence, we are also able to get Olsen-type inequality on the same spaces. Here, the condition of upper Ahlfors n-regular by identity is assumed to obtain the inequalities.

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1. Introduction

Poisson equations play an important role in the field of differential equations and their applications in physics. Closely related to a Poisson equation, we have the fractional integral operator or Riesz potential I_α $(0 < \alpha < d)$, which is defined by:

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\,\mathrm{d}y, \quad x \in \mathbb{R}^d,$$

for suitable functions f on $\mathbb{R}^d$. In Lebesgue spaces over Euclidean spaces, the Riesz potential I_α satisfies the strong inequality:

$$\|I_\alpha f\|_{L^q} \le C_p \|f\|_{L^p},$$

whenever $f \in L^p(\mathbb{R}^d)$ with $1 < p < \frac{d}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$; and also the weak inequality:

$$\left|\{x \in \mathbb{R}^d : |I_\alpha f(x)| > \gamma\}\right| \le C\left(\frac{\|f\|_{L^1}}{\gamma}\right),$$

whenever $f \in L^1(\mathbb{R}^d)$ with $\frac{1}{q} = 1 - \frac{\alpha}{n}$. These inequalities were proved by Hardy and Littlewood [11] and extended later by Sobolev [26].

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Some extensions of Hardy–Littlewood–Sobolev inequalities have been established in Morrey spaces (over Euclidean spaces); see, for examples, [1,3]. For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^d)$ consists of all functions f on $\mathbb{R}^d$ for which:

$$\|f\|_{L^{p,\lambda}} := \sup_{B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(x)|^p \,\mathrm{d}x \right)^{1/p} < \infty.$$

These spaces may be identical to Lebesgue spaces for special cases, namely $L^{p,0} = L^p$ and $L^{p,d} = L^\infty$. Morrey spaces were first introduced by Morrey [14] to study the behavior of solutions to a partial differential equation.

As the Riesz potential I_α is a fractional power of the Laplacian operator, Olsen [18] applied an extension of Hardy–Littlewood–Sobolev inequality in Morrey spaces to study the perturbed Schrödinger operator:

$$-\Delta + V(x) + W(x),$$

where Δ is the Laplacian operator, $V(x)$ is the potential function, and W is a small perturbed potential. Olsen obtained an estimate:

$$\|W I_\alpha f\|_{L^{p,\lambda}} \leq C \|W\|_{L^{(d-\lambda)/\alpha,\lambda}} \|f\|_{L^{p,\lambda}}, \tag{1}$$

for $W \in L^{(d-\lambda)/\alpha,\lambda}$ with $0 \leq \lambda < d - \alpha p$ and $1 < p < \frac{d}{\alpha}$. We will refer to the inequality (1) as the Olsen inequality. Further works on Olsen inequality can be found for examples in [6,9,13,22,24].

Nowadays, various extensions of Hardy–Littlewood–Sobolev inequality can be found in many spaces with different settings—see [4,5,7,8,15–17,19–21,23,25], among others. Particularly, Hajibayov [10] defined the Riesz potential in hypergroups:

$$\begin{aligned} Rf(x) &= \left(\rho(e,r)^{\alpha-n} * f\right)(x) \\ &= \int_K T^x \rho(e,r)^{\alpha-n} f(y^\sim)\,\mathrm{d}\mu(y) \\ &= \int_K \rho(e,r)^{\alpha-n} T^x f(y^\sim)\,\mathrm{d}\mu(y), \end{aligned}$$

and proved the extension of Hardy–Littlewood–Sobolev inequalities (strong and weak inequalities) in Lebesgue spaces over commutative hypergroups. A hypergroup $(K, *)$ is a locally compact Hausdorff space K equipped with a bilinear, associative, and weakly continuous convolution $*$ on $M^b(K)$ (i.e., the set of bounded Radon measure on K) satisfying the following properties:

1. For all $x, y \in K$, the convolution $\delta_x * \delta_y$ of the point measures is a probability measure with compact support.
2. The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ of $K \times K$ into the space of nonempty compact support subsets of K is continuous with respect to the Michael topology.
3. There is an identity $e \in K$, such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
4. There is a continuous involution $*$ (i.e., a homeomorphism $x \mapsto x^\sim$ of K onto itself with the property $(x^\sim)^\sim = x$ for all $x \in K$), such that $\delta_{x^\sim} * \delta_{y^\sim} = (\delta_x * \delta_y)^\sim$.

5. For $x, y \in K$, we have $e \in \mathrm{supp}(\delta_x * \delta_y)$ if only if $x = y^{\sim}$.

(One may see [2,12] for more explanation on hypergroups.) A locally compact Hausdorff group with the group convolution is an example of hypergroup. If

$$\delta_x * \delta_y = \delta_y * \delta_x$$

for every $x, y \in K$, then the hypergroup $(K, *)$ (which is often written just as K) is a commutative hypergroup.

The proof of the extension of Hardy–Littlewood–Sobolev inequalities (strong and weak inequalities) in Lebesgue spaces over commutative hypergroups involves the condition of *upper Ahlfors n-reguler by identity*, namely:

$$\mu(B(e, r)) \leq C r^n \tag{2}$$

for some positive constant which is independent of $r > 0$. Here, e denotes the identity of the hypergroup. The results in this type of Lebesgue spaces assume that the maximal operator satisfies strong and weak inequalities in the Lebesgue spaces under consideration. Here, the maximal operator M is defined by:

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(e,r))} \int_{B(e,r)} T^x |f(y^{\sim})| \, \mathrm{d}\mu(y).$$

As the Hardy–Littlewood–Sobolev inequality in Lebesgue spaces over Euclidean spaces can be extended into Morrey spaces over Euclidean spaces, our aim in this paper is then to extend the results of Hajibayov [10] to Morrey spaces over commutative hypergroups. The proof will not invoke any results on maximal operator in Morrey spaces. Furthermore, we will also prove an Olsen inequality in Morrey spaces over commutative hypergroups.

2. Main Results

For $1 \leq p < \infty$ the Morrey space over commutative hypergroups $L^{p,\lambda}(K) = L^{p,\lambda}(K, *, \mu)$ consists of all measurable functions f on K with norm:

$$\|f\|_{L^{p,\lambda}(K)} := \sup_{B=B(e,r)} \left(\frac{1}{\mu(B(e,2r))^{\lambda/n}} \int_{B(e,r)} |f(y)|^p \, \mathrm{d}\mu(y) \right)^{1/p} < \infty.$$

An extension of Hardy–Littlewood–Sobolev inequality in these spaces is provided in the following theorem.

Theorem 2.1. *Assume that $0 < \lambda < n$, $0 < \theta < n$, $1 < p < \frac{n}{\alpha}$, and the measure μ is upper Ahlfors n-reguler by identity. Assume also that the maximal operator is an operator of strong type-(p,p) on Lebesgue spaces $L^p(K)$. If $\frac{\theta}{q} = \frac{\lambda}{p}$ and $\alpha = \frac{n}{p} - \frac{n}{q}$, then there is a positive constant C, such that the operator R_α satisfies the inequality:*

$$\|R_\alpha f\|_{L^{q,\theta}(K)} \leq C \|f\|_{L^{p,\lambda}(K)}$$

for any function $f \in L^{p,\lambda}(K)$.

Proof. Given $f \in L^{p,\lambda}(K)$, we split it into $f = f_I + f_O := f\chi_{B(e,2r)} + f\chi_{K\setminus B(e,2r)}$. For the function f_I, we have the following estimate:

$$\begin{aligned}\|f_I\|_{L^p(K)} &= \left(\int_K |f_I(x)|^p \, d\mu(x)\right)^{1/p} \\ &= \left(\int_{B(e,2r)} |f(x)|^p \, d\mu(x)\right)^{1/p} \\ &= \frac{(\mu(B(e,2r)))^{\lambda/n}}{(\mu(B(e,r)))^{\lambda/n}} \left(\int_{B(e,2r)} |f(x)|^p \, d\mu(x)\right)^{1/p} \\ &\le (\mu(B(e,2r)))^{\lambda/n} \|f\|_{L^{p,\lambda}(K)}.\end{aligned}$$

Since the maximal operator is an operator of strong type-(p,p) on $L^p(K)$, Hajibayov [10] established that R_α is bounded from $L^p(K)$ to $L^q(K)$. By this boundedness of R_α and the assumption $\frac{\theta}{q} = \frac{\lambda}{p}$, we get:

$$\begin{aligned}&\left(\frac{1}{(\mu(B(e,2r)))^{\theta/n}} \int_{B(e,r)} |R_\alpha f_I(x)|^q \, d\mu(x)\right)^{1/q} \\ &\le \frac{1}{(\mu(B(e,2r)))^{\theta/nq}} \|R_\alpha f_1\|_{L^q(K)} \\ &\le \frac{1}{(\mu(B(e,2r)))^{\theta/nq}} \|f_I\|_{L^p(K)} \\ &\le \frac{C(\mu(B(e,2r)))^{\lambda/np}}{(\mu(B(e,2r)))^{\theta/nq}} \|f\|_{L^{p,\lambda}(K)} \\ &= C\|f\|_{L^{p,\lambda}(K)}.\end{aligned}$$

As a consequence:

$$\|R_\alpha f_I\|_{L^{q,\theta}(K)} \le C\|f\|_{L^{p,\lambda}(K)}.$$

Now, to find the estimate for $\|R_\alpha f_O\|_{L^{q,\lambda}(K)}$, we first need to find an estimate for $R_\alpha f_O$, that is:

$$\begin{aligned}&|R_\alpha f_O(x)| \\ &\le \int_K \frac{|T^x f_2(y^\sim)}{\rho(e,y)^{n-\alpha}} \, d\mu(y) \\ &\le \int_{K\setminus B(e,2r)} \frac{|T^x f_2(y^\sim)}{\rho(e,y)^{n-\alpha}} \, d\mu(y) \\ &\le \sum_{j=1}^{j=\infty} \int_{B(e,2^{j+1}r)\setminus B(e,2^j r)} \frac{|T^x f(y^\sim|)}{\rho(e,y)^{n-\alpha}} \, d\mu(y) \\ &\le \sum_{j=1}^{j=\infty} \frac{1}{(2^j r)^{n-\alpha}} \int_{B(e,2^{j+1}r)} |T^x f(y^\sim)| \, d\mu(y)\end{aligned}$$

$$\leq C \sum_{j=1}^{j=\infty} \frac{\left(\frac{\int_{B(e,2^{j+1}r)} |T^x f(y^\sim)|^p \, \mathrm{d}\mu(y)}{(\mu(B(e,2^{j+2}r)))^{\frac{\lambda}{n}}}\right)^{\frac{1}{p}} \left(\frac{\int_{B(e,2^{j+1}r)} \mathrm{d}\mu(y)}{(\mu(B(e,2^{j+2}r)))^{\frac{\lambda}{n}}}\right)^{1-\frac{1}{p}}}{(2^j r)^{n-\alpha}}$$

$$\leq C \sum_{j=1}^{j=\infty} \frac{1}{(2^j r)^{n-\alpha}} \|f\|_{L^{p,\lambda}(K)} (\mu(B(e,2^{j+2}r)))^{(1-\lambda/n)(1-1/p)}.$$

As μ satisfies upper Ahlfors n-regular by an identity, we get:

$$|R_\alpha f_O(x)| \leq C \sum_{j=1}^{j=\infty} \frac{1}{(2^j r)^{n-\alpha}} \|f\|_{L^{p,\lambda}(K)} (2^{j+2}r)^{n(1-\lambda/n)(1-1/p)}$$

$$= C \sum_{j=1}^{j=\infty} \frac{1}{(2^j r)^{n-\alpha}} \|f\|_{L^{p,\lambda}(K)} (2^{j+2}r)^{(n-\lambda)(1-1/p)}$$

$$= C\|f\|_{L^{p,\lambda}(K)} \sum_{j=1}^{j=\infty} (2^{j+2}r)^{\alpha-\lambda-n/p+\lambda/p}.$$

Since $\alpha = \frac{n}{p} - \frac{n}{q}$, we have:

$$\alpha - \lambda - \frac{n}{p} + \frac{\lambda}{p} = \frac{n}{p} - \frac{n}{q} - \lambda - \frac{n}{p} + \frac{\lambda}{q} = \frac{\lambda q - np}{pq} - \lambda < \frac{\lambda q - np}{pq}.$$

Note also that the assumption $\frac{\theta}{q} = \frac{\lambda}{p}$ and $0 < \theta < n$ enable us to get:

$$\lambda q - np = \theta p - np = (\theta - n)p < 0.$$

Therefore:

$$|R_\alpha f_O(x)| \leq C\|f\|_{L^{p,\lambda}(K)} r^{\frac{\lambda q - np}{pq}} \sum_{j=1}^{j=\infty} 2^{\frac{j(\lambda q - np)}{pq}}$$

$$\leq C r^{\frac{\lambda q - np}{pq}} \|f\|_{L^{p,\lambda}(K)}.$$

We then use this last inequality and apply once more the condition of upper Ahlfors n-regular by an identity to obtain:

$$\left(\frac{1}{(\mu(B(e,2r)))^{\frac{\theta}{n}}} \int_{B(e,r)} |R_\alpha f_O|^q \, \mathrm{d}\mu(x)\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{(\mu(B(e,2r)))^{\frac{\theta}{n}}} \int_{B(e,r)} \left(C r^{\frac{\lambda q - np}{pq}} \|f\|_{L^{p,\lambda}(K)}\right)^q \, \mathrm{d}\mu(x)\right)^{\frac{1}{q}}$$

$$= \frac{C r^{\frac{\lambda q - np}{pq}}}{(\mu(B(e,2r)))^{\frac{\theta}{nq}}} \|f\|_{L^{p,\lambda}(K)} (\mu(B(e,2r)))^{\frac{1}{q}}$$

$$\leq C r^{\frac{\lambda q - np}{pq} + \frac{n}{q} - \frac{\theta}{q}} \|f\|_{L^{p,\lambda}(K)}$$

$$= C r^{\frac{\lambda}{p} - \frac{\theta}{q}} \|f\|_{L^{p,\lambda}(K)}$$

$$= C\|f\|_{L^{p,\lambda}(K)}.$$

This inequality gives us:

$$\|R_\alpha f_O\|_{L^{q,\theta}(K)} \leq C\|f\|_{L^{p,\lambda}(K)},$$

and hence, the desired result follows. □

When, in Theorem 2.1, we have $\theta = \lambda$, then:

$$\|R_\alpha f\|_{L^{q,\lambda}(K)} \leq C\|f\|_{L^{p,\lambda}(K)}. \tag{3}$$

This inequality leads us to the following theorem.

Theorem 2.2. *If* $0 < \lambda < n$, $1 < p < n/\alpha$, *and* $\alpha = \frac{n}{p} - \frac{n}{q}$, *then the inequality:*

$$\mu\left(\{x \in B(e,r) : |R_\alpha f(x)| > \gamma\}\right) \leq C\left(\frac{r^{\lambda/q}\|f\|_{L^{p,\lambda}(K)}}{\gamma}\right)^q$$

holds.

Proof. Let $E_\gamma = (\{x \in B(e,r) : |R_\alpha f(x)| > \gamma\})$. Note that $|R_\alpha f(x)| > \gamma$ gives us $|R_\alpha f(x)|^q > \gamma^q$ for $q > 0$. Hence, $\left(\frac{|R_\alpha f(x)|}{\gamma}\right)^q > 1$. Using the inequality (3), we get:

$$\begin{aligned}
\mu(E_\gamma) &= \int_{E_\gamma} \mathrm{d}\mu(x) \leq \int_{E_\gamma} \left(\frac{|R_\alpha f(x)|}{\gamma}\right)^q \mathrm{d}\mu(x) \\
&\leq \frac{r^{\lambda q}}{\gamma^q}\left[\frac{1}{r^\lambda}\left(\int_{E_\gamma} |R_\alpha f(x)|^q \mathrm{d}\mu(x)\right)^{\frac{1}{q}}\right]^q \\
&\leq \frac{Cr^{\lambda q}}{\gamma^q}\|R_\alpha f\|^q_{L^{q,\lambda}(K)} \\
&\leq \frac{Cr^{\lambda q}}{\gamma^q}\|f\|^q_{L^{p,\lambda}(K)} \\
&= C\left(\frac{r^\lambda\|f\|_{L^{p,\lambda}(K)}}{\gamma}\right)^q,
\end{aligned}$$

which completes our proof. □

Theorem 2.2 provides us with the weak-(p,q) inequality for $1 < p < n/\alpha$. Furthermore, the weak-$(1,q)$ will be presented in the following theorem.

Theorem 2.3. *For* $0 < \alpha + \lambda < n$, *we have:*

$$\mu\left(\{x \in B(e,r) : |R_\alpha f(x)| > \gamma\}\right) \leq C\left(\frac{r^\lambda\|f\|_{L^{1,\lambda}(K)}}{\gamma}\right)^q,$$

provided that $\alpha = n - \frac{n}{q}$.

Proof. When $f \in L^{1,\lambda}(K)$ is decomposed into

$$f = f_I + f_O = f\chi_{B(e,2r)} + f\chi_{K\setminus B(e,2r)},$$

we find that:

$$\begin{aligned}\|f_I\|_{L^1(K)} &\leq (\mu(B(e,2r)))^{\lambda/n}\|f\|_{L^{1,\lambda}(K)}\\ &\leq Cr^{\lambda}\|f\|_{L^{1,\lambda}(K)}.\end{aligned}$$

Hence, by the weak-$(1,q)$ estimate on Lebesgue spaces:

$$\begin{aligned}\mu\left(\{x\in B(e,r): |R_\alpha f_1(x)|>\gamma\}\right) &\leq C\left(\frac{r^{\lambda}\|f\|_{L^1(K)}}{\gamma}\right)^q\\ &\leq C\left(\frac{r^{\lambda}\|f\|_{L^{1,\lambda}(K)}}{\gamma}\right)^q.\end{aligned}$$

Now, for f_O, we use the upper Ahlfors n-regular by an identity to obtain:

$$\begin{aligned}|R_\alpha f_O(x)| &\leq \int_K \frac{T^x f_O(y^{\sim})}{\rho(e,y)^{n-\alpha}}\,\mathrm{d}\mu(y)\\ &\leq \int_{K\setminus B(e,2r)} \frac{T^x f(y^{\sim})}{\rho(e,y)^{n-\alpha}}\,\mathrm{d}\mu(y)\\ &\leq C\sum_{j=1}^{\infty}\int_{2^j r\leq\rho(e,y)<2^{j+1}r}\frac{T_x f(y^{\sim})}{\rho(e,y)^{n-\alpha}}\,\mathrm{d}\mu(y)\\ &\leq C\sum_{j=1}^{\infty}\frac{1}{(2^j r)^{n-\alpha}}\int_{B(e,2^{j+1}r)}|T^x f(y^{\sim})|\,\mathrm{d}\mu(y)\\ &\leq C\sum_{j=1}^{\infty}(2^j r)^{\alpha-n}\left(\mu(B(e,2^{j+2}r))\right)^{\lambda/n}\|T^x f\|_{L^{1,\lambda}(K)}\\ &\leq C\|f\|_{L^{1,\lambda}(K)}\sum_{j=1}^{\infty}(2^j r)^{\alpha+\lambda-n}\\ &\leq Cr^{\alpha+\lambda-n}\|f\|_{L^{1,\lambda}(K)}.\end{aligned}$$

If we choose $\gamma_0=r^{\alpha+\lambda-n}\|f\|_{L^{1,\lambda}(K)}$, then we find that:

$$\begin{aligned}\left(\frac{r^{\lambda}\|f\|_{L^{1,\lambda}(K)}}{\gamma_0}\right)^q &= \left(\frac{r^{\lambda}\|f\|_{L^{1,\lambda}(K)}}{r^{\alpha+\lambda-n}\|f\|_{L^{1,\lambda}(K)}}\right)^q\\ &= \left(Cr^{n-\alpha}\right)^q\\ &= \left(Cr^{n-n-\frac{n}{q}}\right)^q\\ &= Cr^n.\end{aligned}$$

Therefore, for $\gamma_0\leq\gamma$, we have $|R_\alpha f_O|<\gamma_0\leq\gamma$. Consequently:

$$\mu\left(\{x\in B(e,r): |R_\alpha f_O(x)|>\gamma\}\right)=\mu(\emptyset)=0.$$

Meanwhile, for $\gamma>\gamma_0$, we have:

$$\begin{aligned}\mu\left(\{x\in B(e,r): |R_\alpha f_2(x)|>\gamma\}\right) &\leq \mu(B(e,r))\\ &\leq Cr^n\end{aligned}$$

$$= \left(\frac{r^\lambda ||f||_{L^{1,\lambda}(K)}}{\gamma_0} \right)^q$$
$$\leq C \left(\frac{r^\lambda ||f||_{L^{1,\lambda}(K)}}{\gamma} \right)^q .$$

Hence, we are done. □

Having the extension of Hardy–Littlewood–Sobolev inequality in Lebesgue spaces over commutative hypergroups, we could get an Olsen type inequality in these spaces. This inequality is similar to the result in [24] for non-homogeneous type spaces.

Theorem 2.4. *If* $1 < p < \frac{n}{\alpha}$ *and* $\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}$, *then the inequality*

$$||WR_\alpha f||_{L^p(K)} \leq C||W||_{L^{n/\alpha,\lambda}(K)}||f||_{L^p(K)}$$

holds whenever $W \in L^{n/\lambda}(K)$.

Proof. We apply Hölder inequality to get:

$$\left(\int_K |WR_\alpha f(x)^p \, \mathrm{d}\mu(x)| \right)^{\frac{1}{p}}$$
$$\leq \left(\int_K |W(x)|^{\frac{pq}{q-p}} \, \mathrm{d}\mu(x) \right)^{\frac{q-p}{pq}} \left(\int_K |R_\alpha f(x)|^q \, \mathrm{d}\mu(x) \right)^{\frac{1}{q}}$$
$$= \left(\int_K |W(x)|^{\frac{n}{\alpha}} \, \mathrm{d}\mu(x) \right)^{\frac{\alpha}{n}} \left(\int_K |R_\alpha f(x)|^q \, \mathrm{d}\mu(x) \right)^{\frac{1}{q}} .$$

Since R_α is bounded from $L^p(K)$ to $L^q(K)$, we obtain:

$$||WR_\alpha f||_{L^p(K)} \leq C||W||_{L^{n/\alpha,\lambda}(K)}||f||_{L^p(K)},$$

which we wish to prove. □

Now, we are extending the result of Olsen [18] to Morrey spaces over hypergroups.

Theorem 2.5. *If* $0 < \lambda < n$, $1 < p < \frac{n}{\alpha}$, *and* $W \in L^{n/\alpha,\lambda}(K)$, *then the inequality*

$$||WR_\alpha f||_{L^{p,\lambda}(K)} \leq C||W||_{L^{n/\alpha,\lambda}(K)}||f||_{L^{p,\lambda}(K)}$$

holds.

Proof. First, it follows from Hölder inequality that:

$$\left(\int_{B(e,2r)} |WR_\alpha f(x)|^p \mathrm{d}\mu(x) \right)$$
$$\leq \left(\int_{B(e,2r)} |W(x)|^{\frac{pq}{q-p}} \mathrm{d}\mu(x) \right)^{\frac{q-p}{q}} \left(\int_{B(e,2r)} |R_\alpha f(x)|^q \mathrm{d}\mu(x) \right)^{\frac{p}{q}} .$$

As a consequence, we have:

$$\begin{aligned}
&\left(\frac{1}{\mu(B(e,2r)^{\lambda/n}}\int_{B(e,2r)}|WR_\alpha f(x)|^p\mathrm{d}\mu(x)\right)^{1/p}\\
&\quad\le\left(\frac{1}{\mu(B(e,2r)^{\lambda/n}}\int_{B(e,2r)}W(x)^{\frac{pq}{q-p}}\mathrm{d}\mu(x)\right)^{\frac{q-p}{pq}}\\
&\qquad\times\left(\frac{1}{\mu(B(e,2r)^{\lambda/n}}\int_{B(e,2r)}|R_\alpha f(x)|^q\mathrm{d}\mu(x)\right)^{1/q}\\
&\quad\le\left(\frac{1}{\mu(B(e,2r)^{\lambda/n}}\int_{B(e,2r)}|W(x)|^{\frac{n}{\alpha}}\mathrm{d}\mu(x)\right)^{\frac{\alpha}{n}}\\
&\qquad\times\left(\frac{1}{\mu(B(e,2r)^{\lambda/n}}\int_{B(e,2r)}|R_\alpha f(x)|^q\mathrm{d}\mu(x)\right)^{1/q}.
\end{aligned}$$

Now, by applying the inequality (3), we obtain:

$$\begin{aligned}
||WR_\alpha f||_{L^{p,\lambda}(K)}&\le||W||_{L^{n/\alpha,\lambda}(K)}||R_\alpha f||_{L^{q,\lambda}(K)}\\
&\le C||W||_{L^{n/\alpha,\lambda}(K)}||f||_{L^{p,\lambda}(K)},
\end{aligned}$$

which is the desired inequality. □

3. Concluding Remarks

In [25], the Adams-type inequalities have been established on Morrey spaces over metric measure spaces of non-homogeneous type. Typically, the proof of Adams-type inequalities needs some results on the maximal operator in the same spaces. Besides, the results in [25] do not employ any growth condition on measure, which is almost similar to the upper Ahlfors n-regular by an identity condition for measure (Eq. (2)). In this paper, by employing this upper Ahlfors measure, we provide Spanne-type inequalities on Morrey spaces over commutative hypergroups. To prove these inequalities, we do not use any result associated with the maximal operator on Morrey spaces; we only take into account the results of Riesz potential in Lebesgue spaces over commutative hypergroups.

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Idha Sihwaningrum and Sri Maryani
Jenderal Soedirman University
Purwokerto
Indonesia
e-mail: idha.sihwaningrum@unsoed.ac.id

Sri Maryani
e-mail: sri.maryani@unsoed.ac.id

Hendra Gunawan
Bandung Institute of Technology
Bandung
Indonesia
e-mail: hgunawan@math.itb.ac.id

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