The Bivariate Central Normal Distribution

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Abstract. The research studied Probability Density Function (pdf), Cumulative Distribution Function (CDF) and graphical analysis of the bivariate central normal distribution. The CDF of this distribution is then used to compute power of the test Pre-Test-Test (PTT) in the testing intercept. The tables and graphs of the pdf (and CDF) of the bivariate central normal distribution (BCND) and power of the test in testing intercept are produced using R-code. The results showed that the mean and coefficient correlation have a significant affect on the curve, but the variance is not. The curves tend to be leptokurtic for small coefficient correlation, and they will be mesocurtic for large coefficient correlation. The power of the PTT increases as the CDF of the BCND increases.

1. Introduction

The normal (Gaussian) distribution is the most popular distribution. It often uses in many areas and statistical analysis [3]. Probability Density Function (PDF) of this distribution (univariate normal distribution) of X random variable with mean (μ) and variance (σ^2), $X \sim N(\mu, \sigma^2)$, is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, \text{ with } -\infty < \mu < \infty \text{ and } 0 < \sigma < \infty. \tag{1}$$

The values of the pdf of the equation (1) have not seen in textbooks yet. Currently, the availability table of statistics of this distribution is a normal standard distribution table for $\mu = 0$ the end $\sigma^2 = 1$.

From the equation (1), it is clear that the computational of the probability integral of the pdf and cumulative distribution function *(CDF)* are very complicated and hard, so they should be numerically computed [8]. Due to the values of the pdf and CDF are critical values that use to accept or reject the null hypothesis (H_0), then we need to create the statistics table for both (the pdf and CDF).

Moreover, there are several problems in statistical analysis in term of two or more of random variables. In this case, all the computational of the pdf and CDF are very complicated and challenging. It is because of distribution is a joint distribution of the normal random variable X and Y. Let ρ is a coefficient correlation between them, then if $\rho \neq 0$, the bivariate central normal distribution (bivariate normal distribution) of X and Y is the joint distribution (multiple) of the marginal and conditional normal distribution [16]. Furthermore, Sahoo [6] has also studied and presented the pdf of the central bivariate normal distribution (BCND) of X and Y, that is

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\left(\left[\frac{x-\mu_1}{\sigma_1}\right]^2 - 2\rho\left[\frac{x-\mu_1}{\sigma_1}\right]\left[\frac{y-\mu_2}{\sigma_2}\right] + \left[\frac{y-\mu_2}{\sigma_2}\right]^2\right)\right)}, \quad -\infty < x, y < \infty \tag{2}$$

with $\mu_1, \mu_2 \in \square$, $\sigma_1, \sigma_2 \in (0,\infty)$, and $\rho \in (-1,1)$. The notation of the BCND is then written as

$$(X,Y) \square\ N\left(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho\right).$$

Many authors already used the BCND, such as Pratikno [2] and Khan [10] in testing intercept using *non-sample prior information* (NSPI). Here, Pratikno [2] used the BCND to compute power of the test of *pre test-test* (PTT) in testing intercept using NSPI on simple regression model, while Khan [9,10], Khan and Saleh [11,13,14], Khan and Hoque[12], Khan and Pratikno [15], Saleh [1] and Yunus [7] contributed to developing the research in estimation area. All the authors used *R-code* and *R-package* especially **mvtnorm** on the *multivariate (bivariate/BCND) normal distribution*.

Due to the importance of the BCND, we studied more detail of this distribution particularly in computational of the PDF, CDF, graphical analysis and its applied in NSPI testing. The steps of the research methodology are (1) reconstructed and figured the pdf formula, (2) computed CDF table and (3) graphical analysis. *R-code* is used to figure pdf and compute the table of the CDF of the BCND.

The research presented the introduction in Section 1. Reconstructed the pdf formula of the BCND is obtained in Section 2. The graphs of the pdf of BCND and its analysis are given in Section 3. Section 4 explained the computational of the values of the CDF of the BCND. Furthermore, computational of the power of the test is obtained in Section 5, and Section 6 described conclusion and remark.

2. Method

Following Kenny and Keeping [5], we reconstructed the formula of the equation (2) in term of two independent random variables of the univariate normal standard, Z_1 and Z_2, namely
$$X = \mu_1 + \sigma_{11}Z_1 + \sigma_{12}Z_2 \text{ and } Y = \mu_2 + \sigma_{21}Z_1 + \sigma_{22}Z_2 \tag{3}$$

where X and Y are random variables of the normal distribution with mean (μ_1, μ_2), -variance $(\sigma_1^2, \sigma_2^2) = (\sigma_{11}^2 + \sigma_{12}^2, \sigma_{21}^2 + \sigma_{22}^2)$, and covariance $= \rho\sigma_1\sigma_2 = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}$. Moreover, the joint pdf of Z_1 and Z_2 are then written as $f(z_1,z_2) = f(z_1)f(z_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_1^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_2^2} = \frac{1}{2\pi}e^{-\frac{1}{2}(z_1^2+z_2^2)}$. Using matrix properties and a simple modification, we got the equation in matrix form as below

$$\begin{pmatrix} X-\mu_1 \\ Y-\mu_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ or } \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}^{-1}\begin{pmatrix} X-\mu_1 \\ Y-\mu_2 \end{pmatrix}. \tag{4}$$

From the equation (4), we get the squareof Z_1 and Z_2 on equation (5) below

$$\left(Z_1^2 + Z_2^2\right)\left(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}\right)^2 = \left[\sigma_{22}(X-\mu_1) - \sigma_{12}(Y-\mu_2)\right]^2 + \left[-\sigma_{21}(X-\mu_1) + \sigma_{11}(Y-\mu_2)\right]^2$$

$$= \sigma_1^2\sigma_2^2\left(\left[\frac{X-\mu_1}{\sigma_1^2}\right]^2 - 2\rho\left[\frac{X-\mu_1}{\sigma_1}\right]\left[\frac{Y-\mu_2}{\sigma_2}\right] + \left[\frac{Y-\mu_2}{\sigma_2^2}\right]^2\right), \tag{5}$$

if $\dfrac{1}{1-\rho^2} = \dfrac{\sigma_1^2\sigma_2^2}{\left(\sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}\right)^2}$, then the equation (5) is written as

$$\left(Z_1^2 + Z_2^2\right)(1-\rho)^2 = \left(\left[\frac{X-\mu_1}{\sigma_1^2}\right]^2 - 2\rho\left[\frac{X-\mu_1}{\sigma_1}\right]\left[\frac{Y-\mu_2}{\sigma_2}\right] + \left[\frac{Y-\mu_2}{\sigma_2^2}\right]^2\right).$$

Furthermore, we need Jakobian transformation for setting the joint distribution of X and Y. The Jacobian transformation is then expressed as $J = \dfrac{1}{\sigma_1 \sigma_2 \sqrt{1-\rho^2}}$, and the joint pdf of the X and Y is given by

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(z_1^2 + z_2^2\right)} = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2}Q(x,y)}$$

(6)

where $Q(x,y) = \dfrac{1}{1-\rho^2}\left[\dfrac{(x-\mu_1)^2}{\sigma_1^2} - \dfrac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \dfrac{(y-\mu_2)^2}{\sigma_2^2}\right]$. Due to the complicated formula on the equation (6), *R-code* is then used to compute the values of the pdf of the equation (6).

3. Result and Discussion

The pdf graph of the equation (2) is made by *R-code*. Here, some simulations of the graphs are presented in Figure 1. However, we did not present all the graphs in this paper. Figure 1 showed that ρ has a significant affect on the form of the curve. The curve tends to be leptokurtic for $\rho = -0.9$, and it will be symmetric (mesokurtic) on $\rho = 0$. We note that for $\rho(\rho < 0)$ the curve tends to be leptocurtic, and it will be mesokurtic for $\rho(\rho > 0)$. From several simulations, we see that the curve is concave-convex with $\mu_1 = \mu_2 = 0$, $\rho = 0$, $\sigma_1^2 = \sigma_2^2$ and tends to be mesokurtic for large σ^2 and leptocurtic for small σ^2. Conversely, the curve will be convex concave for $\mu_1 = \mu_2 = 0$; $\rho = 0$, $\sigma_1^2 \neq \sigma_2^2$ However, a parameter σ^2 is not significant in changing the form of the curve.
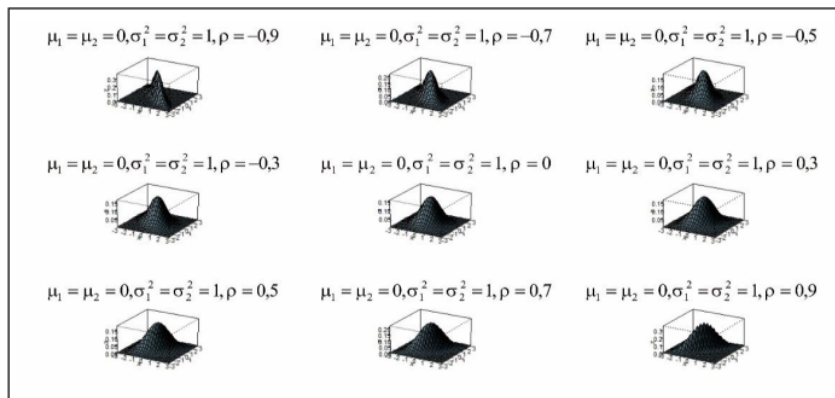
Figure 1. The pdf of the bivariate central normal distribution

To compute the values of the CDF of the bivariate central normal distribution (BCND), *R-code* is used. This is due to the process of its integral is very complicated and difficult. Here, we used *R-command adaptIntegrate* and *package cubature*, and we wrote (run): *adaptIntegrate(f,lowerLimit=c(-100,100), upper Limit=c(x,y))$integral* with lower bound (-100,-100) and up bounds (*x , y*) to produce the CDF as well as the power of the test of the *pre-test test* (PTT). Table 1-3 below are the values of the fCDF of the BCND for some selected ρ.

Table 1. The values of CDF of the BNCD with $\rho = 0.9$

Y	X						
	-1,5	-1,0	-0,5	0,0	0,5	1,0	1,5
-1,5	0,00000	0,00000	0,00000	0,00001	0,00048	0,00535	0,02286
-1,0	0,00000	0,00000	0,00002	0,00070	0,00898	0,04316	0,09720
-0,5	0,00000	0,00002	0,00080	0,01163	0,06321	0,15886	0,24221
0,0	0,00001	0,00070	0,01163	0,07178	0,20309	0,34205	0,43321
0,5	0,00048	0,00898	0,06321	0,20309	0,38373	0,53283	0,62466
1,0	0,00535	0,04316	0,15886	0,34205	0,53283	0,68269	0,77454
1,5	0,02286	0,09720	0,24221	0,43321	0,62466	0,77454	0,86639

Table 2. The values of CDF of the BNCD with $\rho = 0.5$

Y	X						
	-1,5	-1,0	-0,5	0,0	0,5	1,0	1,5
-1,5	0,01832	0,03242	0,04684	0,05765	0,06357	0,06594	0,06663
-1,0	0,03242	0,06251	0,09748	0,12740	0,14621	0,15487	0,15779
-0,5	0,04684	0,09748	0,16332	0,22688	0,27224	0,29609	0,30530
0,0	0,05765	0,12740	0,22688	0,33333	0,41834	0,46874	0,49084
0,5	0,06357	0,14621	0,27224	0,41834	0,54624	0,63028	0,67149
1,0	0,06594	0,15487	0,29609	0,46874	0,63028	0,74520	0,80696
1,5	0,06663	0,15779	0,30530	0,49084	0,67149	0,80696	0,88471

Table 3. The values of CDF of the BNCD with $\rho = 0.0$

| Y	X						
	-1,5	-1,0	-0,5	0,0	0,5	1,0	1,5
-1,5	0,00446	0,01060	0,02061	**0,03340**	0,04619	0,05621	0,06234
-1,0	0,01060	0,02517	0,04895	0,07933	0,10970	0,13348	0,14806
-0,5	0,02061	0,04895	0,09520	0,15427	0,21334	0,25959	0,28793
0,0	0,03340	0,07933	0,15427	0,25000	0,34573	0,42067	0,46660
0,5	0,04619	0,10970	0,21334	0,34573	0,47812	0,58176	0,64527
1,0	0,05621	0,13348	0,25959	0,42067	0,58176	0,70786	0,78514
1,5	0,06234	0,14806	0,28793	0,46660	0,64527	0,78514	0,87085

The Table 1-3 showed that the values of the CDF is going to zero when X and Y decrease (going to be negative), and it increases to 1 as the X and Y increase (going to be positive). Here, the values of the table (CDF) are similar (the same as) for positive and negative correlation (ρ).

Following Pratikno [2] and Khan [9], there are three tests in *testing intercept* using *non-sample prior information* (NSPI), namely *unrestricted test* (UT), *restricted test* (RT) and *pre-test test* (PTT). In this case, the bivariate central normal distribution is used to compute the power of the *pre-test test* (PTT). Here, the power is a probability of reject H_0 ing under $H_a : \theta = \theta_a$, then it is written as

$\pi(\theta_a) = P(reject\ H_0 | \theta = \theta_a)$ (Wackerly et al., 2008, p455). Following Pratikno (2012), the power of the PTT in testing hypothesis *one-side maximum on simple regression model* is given by

$$\pi^{PTT}\left(\lambda_1,\lambda_2\right)=\phi\left(z_{\alpha_3}-\frac{\lambda_2\sqrt{S_{xx}}}{\sigma\sqrt{n}}\right)\left(1-\phi\left(z_{\alpha_2}-\frac{\lambda_1+\lambda_2\bar{X}}{\sigma}\right)\right)$$

$$=+c_{1\rho}\left(z_{\alpha_3}-\frac{\lambda_2\sqrt{S_{xx}}}{\sigma\sqrt{n}},z_{\alpha_1}-\frac{\lambda_1}{m_1},\rho\neq0\right),$$

Where $m_1=\sigma\left[1+\frac{n\bar{X}^2}{S_{xx}}\right]^{1/2}$, $\phi(x)$ is cumulative distribution function (CDF) of the normal distribution and $c_{1\rho}$ is probability integral of the normal bivariate distribution? The graphs of the power of the PTT are presented in Figure 2.
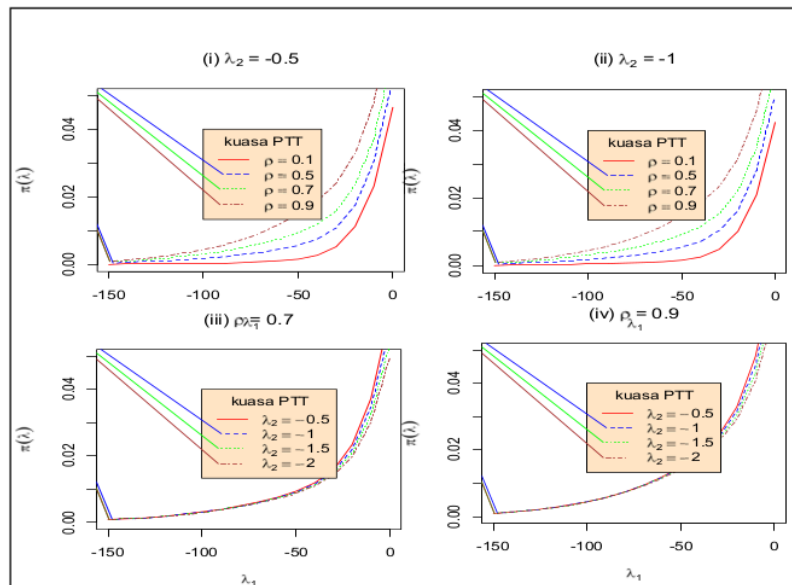
Figure 2. Power of the PTT versus λ_2

From Figure 2 (i) and (ii), and (iii) and (iv), it is clear that the power of the PTT increases as the ρ increases, and it decreases as the λ_2 decreases. It means that the ρ is significant to increase the power of the PTT.

4. Conclusion

The research studied pdf, CDF, graphical analysis of the BCND, and its application on the power of the PTT. To compute the power of the PTT in testing intercept and produce the graphs of the pdf and table CDF, *R-code* is then used. The results showed that the mean and coefficient correlation have a significant affect to the curves, but the variance is not. The curves tend to be leptocurtic for small coefficient correlation, and they will be mesokurtic for large coefficient correlation. On the other hand, the curve will be *leptocurtic* and or *mesokurtic* when the coefficient correlation decreases (and or increases). For both μ_1 and μ_2 are negative, the curve is *skew positive*, and vise verse.

In term of the power of the PTT, the BCND is used to compute the power of the PTT, and the result showed that it increases as the coefficient correlation increases.

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