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Necessary Conditions for a Norm Estimate of Riesz Potential on Morrey Spaces over Hypergroups

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Abstract. Necessary condition for a norm estimate of Riesz potential will be presented on Morrey spaces over commutative hypergroups by taking into account the upper Ahlfors condition. This norm estimate is the Hedberg type estimate. By assuming that the weak estimate of maximal operator holds in Morrey spaces over commutative hypergroups, the Hedberg type estimate leads to the weak estimate of the Riesz potential.

INTRODUCTION

Some estimates regarding Riesz potential, which is also known as fractional integral operator, have been known for many types of spaces. The well known results on strong and weak type estimates for Riesz potential on Lebesgue spaces over Euclidean spaces were provided by Hardy and Littlewood [1] as well as Sobolev [2]. These results were then extended by Petree in Spanne [3], Adams [4], as well as Chiarenza and Frasca [5] into Morrey spaces over Euclidean spaces; and by Nakai [6] as well as Guliyev [7] into generalized Morrey spaces over Euclidean spaces. After Nazarov *et. al* [8] introduced the notion of non-doubling measure, some estimates on Riesz potential were then established on Lebesgue, Morrey, and generalized Morrey spaces over Euclidean spaces with non-doubling measure. (See [9], [10], [11], [12] and some other references.) Later, the estimates of fractional integral operators on Lebesgue spaces, Morrey spaces and their generalization over metric spaces were given under doubling as well as non doubling measure. These can be seen for example in [13], [14],[15], [16], [17]. The estimates can be proved using several norm estimates. One of them is introduced by Hedberg [18] so that some researchers call such norm estimate as Hedberg estimate. Recently, on commutative hypergroups $(K, *)$ which posses a Haar measure μ, Hajibayov [19] defined the Riesz potential I_α to be

$$I_\alpha f(x) = \int_K \rho(e,r)^{\alpha-n} T^x f(f(y^*))\, d\mu(y).$$

Here, T^x (for $x \in K$) denotes the generalized translation operator and is given by

$$T^x f(y) = \int_K f\, d(\delta_x * \delta_y), \quad \forall y \in K,$$

where δ_x and δ_y denote probability measures for x and y respectively. Hajibayov proved that the Riesz potential satisfied the norm estimates, i.e. the strong and weak estimates, for $1 \leq p \leq q < \infty$ on Lebesgue spaces over commutatives hypergroup under the condition of upper Ahlfors n-regular by an identity. This condition says that there exists a positive constant C (which is independent of $r > 0$) such that

$$\mu(B(e,r)) \leq Cr^n.$$

Here the ball $B(e,r)$ has center e (that is the identity of the hypergroup $(K,*)$) and radius r. Note that in this paper, we will denote C as the positive constants which have different values. The results in [19] assume that the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(e,r))} \int_{B(e,r)} T^x|f(y^*|\, d\mu(y)$$

satisfies strong and weak estimates. As it is described previously that the results on Lebesgue spaces can be extended into Morrey spaces, our aim in this paper is then to extend the Hedberg estimate into Morrey spaces over commutative hypergroup; and we will use this Hedberg type estimate to prove the weak estimate of the Riesz potential in the space under consideration. For $1 \leq p \leq q < \infty$, Morrey spaces $\mathcal{M}^{p,q}(K,\mu)$ consist of all μ-measurable functions $f : K \to (-\infty, \infty)$ such that the norm

$$\|f\|_{\mathcal{M}^{p,q}(K,\mu)} := \sup_{B=B(e,r)} \mu(B)^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(x)|^p \, d\mu(x) \right)$$

is finite.

MAIN RESULTS

The Hedberg type estimate will be stated in the following theorem.

Theorem 1. *Let $1 < q < \infty$, $1 < s < \infty$, $0 < \alpha < \frac{\alpha}{q}$. Assume that the measure μ satisfies the condition of upper Ahlfors n-regular by an identity. If $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$, then there is a positive constant C such that the norm estimate*

$$|I_\alpha f(x)| \leq CMf(x)^{q/s} \|f\|^{\alpha q/n}_{\mathcal{M}^{1,q}(K,\mu)}.$$

holds.

Proof. For every f in Morrey space $\mathcal{B}^{1,q}(K,\mu)$, we write

$$I_\alpha f(x) = U_1 + U_2$$

where

$$U_1 = \int_{B(e,r)} \rho(e,r)^{\alpha-n} T^x f(y^*) \, d\mu(y)$$

and

$$U_2 = \int_{K \setminus B(e,r)} \rho(e,r)^{\alpha-n} T^x f(y^*) \, d\mu(y)$$

Firstly, we find an estimate for U_1, which is given by

$$
\begin{aligned}
|U_1| &\le \int_{B(e,r)} \rho^{\alpha-n} T^x |f(y^*)|\, d\mu(y) \\
&\le \sum_{j=1}^{\infty} \int_{2^{-j}r \le \rho(e,r) < 2^{-j+1}r} \frac{T^x|f(y^*)|}{\rho^{n-\alpha}}\, d\mu(y) \\
&\le \sum_{j=1}^{\infty} \frac{1}{(2^{-j}r)^{n-\alpha}} \int_{B(e,2^{-j+1}r)}^{x} |f(y^*)|\, d\mu(y) \\
&= \sum_{j=1}^{\infty} \frac{1}{(2^{-j}r)^{n-\alpha}} \frac{\mu(B(e,2^{-j+1}r))}{\mu(B(e,2^{-j+1}r))} \int_{B(e,2^{-j+1}r)} |f(y^*)|\, d\mu(y) \\
&\le C \sum_{j=1}^{\infty} \frac{(2^{-j+1}r)^n}{(2^{-j}r)^{n-\alpha}} Mf(x) \\
&= C r^{\alpha} Mf(x) \sum_{j=1}^{\infty} 2^{-j\alpha} = C r^{\alpha} Mf(x).
\end{aligned}
$$

Next, we find the estimate for U_2, that is

$$
\begin{aligned}
|U_2| &\le \int_{K\setminus B(e,r)} \rho^{\alpha-n} |T^x f(y^*)|\, d\mu(y) \\
&\le \sum_{j=0}^{\infty} \int_{2^{j}r \le \rho(e,r) < 2^{j+1}r} \frac{|T^x f(y^*)|}{\rho^{n-\alpha}}\, d\mu(y) \\
&\le \sum_{j=0}^{\infty} \frac{1}{(2^{j}r)^{n-\alpha}} \int_{B(e,2^{j+1}r)}^{x} |T^x f(y^*)|\, d\mu(y) \\
&= \sum_{j=0}^{\infty} \frac{1}{(2^{j}r)^{n-\alpha}} \frac{\mu(B(e,2^{-j+1}r))^{1/q-1}}{\mu(B(e,2^{-j+1}r))^{1/q-1}} \int_{B(e,2^{j+1}r)} |T^x f(y^*)|\, d\mu(y) \\
&\le C \sum_{j=0}^{\infty} \frac{1}{(2^{j}r)^{n-\alpha}(2^{j+1}r)^{n(1/q-1)}} \|T^x f\|_{\mathcal{M}^{1,q}(K,\mu)} \\
&\le C r^{\alpha-n/q} \|f\|_{\mathcal{M}^{1,q}(K,\mu)} \sum_{j=0}^{\infty} 2^{j(\alpha-n/q)} \\
&= C r^{\alpha-n/q} \|f\|_{\mathcal{M}^{1,q}(K,\mu)}.
\end{aligned}
$$

By taking

$$
r = \left(\frac{Mf(x)}{\|f\|_{\mathcal{M}^{1,q}(K,\mu)}} \right)^{-q/n},
$$

the following result

$$
\begin{aligned}
|I_\alpha f(x)| &\le C \left(r^{\alpha} Mf(x) + r^{\alpha-n/q} \|f\|_{\mathcal{M}^{1,q}(K,\mu)} \right) \\
&= C Mf(x)^{(n-\alpha q)/n} \|f\|_{\mathcal{M}^{1,q}(K,\mu)}^{\alpha q/n} \\
&= C Mf(x)^{q/s} \|f\|_{\mathcal{M}^{1,q}(K,\mu)}^{\alpha q/n}.
\end{aligned}
$$

follows. □

The Hedberg type estimates then gives us the following theorem.

Theorem 2. *Let $1 < q < \infty$, $1 < s < \infty$, and $0 < \alpha < \frac{\alpha}{q}$. Assume that the measure μ satisfies the condition of upper Ahlfors n-regular by an identity. Assume also that there is a positive constant C_1 such that the maximal operator M satisfies the inequality*

$$\mu\left(\{x \in B(e,r) : Mf(x) > \lambda_1\}\right) \leq \frac{C_1 r^{n(1-1/q)}}{\lambda_1} \|f\|_{\mathcal{M}^{1,q}}(K, \mu.$$

If $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a positive constant C such that

$$\mu\left(\{x \in B(e,r) : I_\alpha f(x) > \lambda\}\right) \leq C r^{n(1-1/q)} \left(\frac{\|f\|_{\mathcal{M}^{1,q}(K,\mu)}}{\lambda}\right)^{s/q}.$$

holds for any positive μ-measurable function f.

Proof. From the Hedberg type estimate in Theorem 1, for $|I_\alpha f(x)| > \lambda$, we have

$$Mf(x) > \left(\frac{\lambda}{\|f\|^{\alpha q/n}_{\mathcal{M}^{1,q}(K,\mu)}}\right)^{s/q}.$$

This last equation and the estimate of the maximal operator then provide us with

$$\begin{aligned}
&\mu\left(\{x \in B(e,r) : I_\alpha f(x) > \lambda\}\right) \\
&\quad \leq \mu\left(\left\{x \in B(e,r) : Mf(x) > \left(\frac{\lambda}{\|f\|^{\alpha q/n}_{\mathcal{M}^{1,q}(K,\mu)}}\right)^{s/q}\right\}\right) \\
&\quad \leq \frac{C r^{n(1-1/q)} \|f\|_{\mathcal{M}^{1,q}(K,\mu)}}{\left(\frac{\lambda}{\|f\|^{\alpha q/n}_{\mathcal{M}^{1,q}(K,\mu)}}\right)^{s/q}} \\
&\quad = C r^{n(1-1/q)} \left(\frac{\|f\|_{\mathcal{M}^{1,q}(K,\mu)}}{\lambda}\right)^{s/q}.
\end{aligned}$$

Thus, the desired estimate is proved. □

CONCLUSION

Here we apply the Hedberg type estimate to find the weak type estimate for Riesz potential. However, we may prove the weak estimate without Hedberg type estimate if we have no assumption on maximal operator.

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