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THE NONCENTRAL *t* DISTRIBUTION AND ITS APPLICATION ON THE POWER OF THE TESTS

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Abstract

We study the noncentral t (NCT) distribution, analyze it graphically and provide its application on the power of the tests in multivariate simple regression model (MSRM). The power of the tests, namely unrestricted test (UT), restricted test (RT) and pre test-test (PTT), are used to test the coefficients of parameters of the MSRM. In computing the power of the tests, plot of their graphs, and the values of the cdf of the NCT distribution and their graphs, R-code is used. The results showed that the curve of the NCT distribution is influenced by the noncentral parameter (δ), $\delta \geq 1$, and the power of the test of the PTT still remains an alternative choice of the tests among UT and RT.

1. Introduction

Following Amos [4], the noncentral t (NCT) distribution is written as $T'_{\nu}(\delta)$ with noncentral parameter (δ), $\delta > 0$, and ν degrees of freedom. For $\delta = 0$, it will be a central t (univariate central t) distribution. The central t distribution is used in many areas of statistical analysis [11] such as testing

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mean (Bain and Engelhardt [11], Hogg and Tanis [18], Larson [8]). We note that the critical values of t central distribution is available in many textbooks. Furthermore, the probability density function (pdf) of central t distribution of a T random variable with ν degrees of freedom, zero mean and variance $\frac{\nu}{\nu-2}$, $\nu \geq 2$, is then given by

$$f(t;\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}}\left[1+\frac{t^2}{\nu}\right]^{-\frac{\nu+1}{2}}, \tag{1}$$

with $Z \sim N(0, 1)$, $U \sim \chi^2(\nu)$ and $T = \frac{Z}{\sqrt{U/\nu}}$.

Moreover, Levy and Narula [10], Johnson et al. [13] and Shao [9] presented the pdf of the NCT distribution in their papers. Some applications of the NCT distribution are found in Johnson and Welch [12], and Cousineau and Laurencelle [6]. Following Levy and Narula [10], the pdf formula of the NCT distribution is given as

$$f(t) = c\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\nu+k+1}{2}\right)\delta^k 2^{\frac{k}{2}} t^k}{\Gamma(k+1)(\nu+t^2)^{\frac{k}{2}}}, \tag{2}$$

where $T'_\nu(\delta) = \frac{Z+\delta}{\sqrt{U/\nu}}$, $Z \sim N(0, 1)$, δ is noncentral parameter, $U \sim \chi^2_\nu$, and ν is degrees of freedom. Here,

$$c = \frac{\nu^{\frac{\nu}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \times \frac{e^{\frac{-\delta^2}{2}}}{(\nu+t^2)^{\frac{(\nu+1)}{2}}}, \quad E[T'_\nu(\delta)] = \delta\left(\frac{\nu}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$$

and

$$Var(T'_\nu(\delta)) = \frac{\nu}{\nu-2}(1-\delta^2) - (E[T'_\nu(\delta)])^2,$$

respectively. The pdf values of equation (2) have not been seen in textbooks yet. Currently, the table of the statistics critical values of this distribution is only for the central *t* distribution $(\delta = 0)$. Note that equation (2) is the pdf default of the NCT distribution that is available in R package. From equation (2), it is clear that the computation of the probability integral of the pdf and cumulative distribution function (cdf) of the NCT distribution are very complicated and hard, so they should be numerically computed, and R-code is then used. As the values of the cdf are used to accept or reject null hypothesis (H_0), we need to create the statistics critical values table for both (the pdf and cdf) of the NCT distribution.

Due to the equivalency of the *F* central distribution with square central *t* distribution, we consider the equivalency of the square noncentral *t* with the noncentral *F* (or bivariate noncentral *F*) distribution (Pratikno [3]). Here, many authors have already studied noncentral *F* distribution, such as Krishnaiah and Armitage [14], Amos and Bulgren [5], Schuurmann et al. [7], El-Bassiouny and Jones [1] and Pratikno [3].

Many authors have used the noncentral (univariate and or bivariate) *F* distribution such as Pratikno [3], Khan and Pratikno [19] and Khan [20] in testing intercept using non-sample prior information (NSPI). Pratikno [3] used the bivariate noncentral *F* (BNCF) distribution to compute the power of the tests of the unrestricted test (UT), restricted test (RT) and pre-test test (PTT) in testing intercept using NSPI on simple regression model (SRM), while Khan [20, 21], Khan and Saleh [23-25], Khan and Hoque [22], Saleh [2] and Yunus [16] contributed to develop the research in estimation area. All the authors have used R-code for calculating the values of the power of the test (PTT).

Because of the importance of this distribution in SRM, we studied this distribution in more detail particularly in computation of the pdf, cdf, graphical analysis and applied in testing intercept on the multivariate simple regression model (MSRM). The steps of the research methodology are (1) re-expressed and figured the pdf formula, (2) computed the values of the cdf of the NCT distribution, and (3) graphical analysis of the power of the tests.

Here, *R-code* is used to figure pdf and compute the values of the cdf of the NCT distribution and power of the tests (UT, RT and PTT).

Analysis of the noncentral *t* distribution is given in Section 2. The power and size of the tests are obtained in Section 3. Section 4 describes the conclusion of the research.

2. Analysis of the Noncentral *t* Distribution

In this section, we re-expressed the pdf formula of equation (2) as follows. Let $Z \sim N(0, 1)$ and $U \sim \chi_\nu^2$, and $T'_\nu(\delta)$ be a noncentral random variable given by $T = \frac{Z + \delta}{\sqrt{U/\nu}}$, with $Z = z$, $U = u$, $T'_\nu(\delta) = t$ and $Z = T\sqrt{U/\nu} - \delta$. Then the join pdf of the Z and U is given as

$$f(z, u) = f(z) \cdot f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}}.$$

Furthermore, the join pdf of the noncentral random variable T and U is given as

$$\begin{aligned} f(t, u) &= f(t\sqrt{u/\nu} - \delta, u) \cdot |J| \\ &= \frac{1}{\sqrt{\pi}\, 2^{\nu+1/2}\Gamma\left(\frac{\nu}{2}\right)} u^{\frac{\nu}{2}-1} e^{-\frac{(t\sqrt{u/\nu}-\delta)^2}{2} - \frac{u}{2}} \cdot \sqrt{\frac{u}{\nu}}. \end{aligned}$$

The marginal function of $f(t)$ is then obtained as

$$\begin{aligned} f(t) &= \int_0^\infty f(t, u)\, du = \int_0^\infty \frac{2^{-(\nu+1/2)}\nu^{-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} u^{\frac{\nu}{2}-\frac{1}{2}} e^{-\frac{1}{2}[(t\sqrt{u/\nu}-\delta)^2+u]} du \\ &= \frac{2^{-(\nu+1/2)}\nu^{-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} e^{-\frac{\delta^2}{2}} \sum_{k=0}^{\infty} \frac{\delta^k t^k}{\nu^{k/2}\Gamma(k+1)} \int_0^\infty u^{\frac{\nu+k-1}{2}} e^{-\frac{1}{2}\left[u\left(\frac{t^2}{\nu}+1\right)\right]} du, \quad (3) \end{aligned}$$

with $e^{\delta t\sqrt{u/\nu}} = \sum_{k=0}^{\infty} \frac{\delta^k t^k u^{k/2}}{\nu^{k/2}\Gamma(k+1)}$ and $\Gamma(k) = (k-1)!$. Equation (3) is written as

$$f(t) = \frac{2^{-(\nu+1/2)}\nu^{-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} e^{-\frac{\delta^2}{2}}$$

$$\times \sum_{k=0}^{\infty} \frac{\delta^k t^k \nu^{\frac{\nu+1}{2}}}{\Gamma(k+1)(t^2+\nu)^{\frac{\nu+k+1}{2}}} \int_0^{\infty} w^{\frac{\nu+k-1}{2}} e^{-\frac{1}{2}w} dw,$$

with $du = \frac{\nu}{t^2+\nu} dw$ and $u = \frac{vw}{(t^2+\nu)}$. Applying $\beta^{\alpha}\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$ to the above equation, we obtain

$$f(t) = \frac{2^{-(\nu+1/2)}\nu^{-\frac{1}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} e^{-\frac{\delta^2}{2}}$$

$$\times \sum_{k=0}^{\infty} \frac{\delta^k t^k \nu^{\frac{\nu+1}{2}}}{\Gamma(k+1)(t^2+\nu)^{\frac{\nu+k+1}{2}}} 2^{\frac{\nu+k+1}{2}} \Gamma\left(\frac{\nu+k+1}{2}\right)$$

$$= c \times \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\nu+k+1}{2}\right)}{\Gamma(k+1)} \frac{\delta^k t^k 2^{\frac{k}{2}}}{(t^2+\nu)^{\frac{k}{2}}} \tag{4}$$

with $c = \frac{\nu^{\frac{\nu}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \times \frac{e^{-\frac{\delta^2}{2}}}{(t^2+\nu)^{\frac{\nu+1}{2}}}$.

The graphs of the pdf of the NCT distribution of equation (4) for some selected δ, $\delta = 0, 1, 2, ..., 5$, at $\nu = 10$ and $k = 110$, are given in Figure 1.

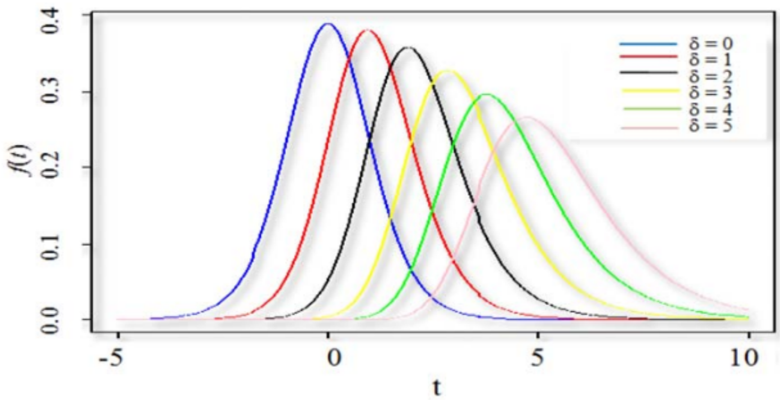

Figure 1. pdf plot of the NCT distribution for $\nu = 10$, $\delta = [0, 1, 2, 3, 4, 5]$, $k = 110$.

From Figure 1, we see that the curves change to the right as δ increases. However, the form of the curves does not significantly change, but tends to be skew positive later. It is also clear that the centers (extreme) of the curves decline with increase in δ $(\delta \geq 1)$.

A simulation study is given to produce the values of the cdf of the noncentral t distribution for $\nu = 10$, $\delta = 3$ and $k = 110$. These are presented in Table 1.

Table 1. The values of the cdf of the NCT distribution at $k = 110$

t	–3	–2	–1	0	1	2	3
The values of the cdf, $k = 110$	0.002	0.022	0.139	0.460	0.812	0.966	0.996

Note that k has no significant effect to the values of the pdf of the NCT distribution. The values of the cdf are computed using R. Here, the lower limit of the integral is not infinity. This is due to the fact that R does not work at infinity, it works at –100 as minimum lower limit. Due to this obstacle, we simulate **to produce** the critical values of the *NCT* distribution for some selected α, δ and ν in Table 2.

Table 2. The critical values of the NCT distribution for $\delta = 0.01$ and 0.05; $\nu = 5, 10, 15$ at $\alpha = 0.01$ and 0.05

$\alpha = 0.01$

δ	$1-\alpha$	$\nu = 5$	$\nu = 10$	$\nu = 15$
0.01	0.99	3.38	2.78	2.61
0.05	0.99	3.46	2.83	2.66

$\alpha = 0.05$

δ	$1-\alpha$	$\nu = 5$	$\nu = 10$	$\nu = 15$
0.01	0.95	2.03	1.82	1.76
0.05	0.95	2.09	1.87	1.81

3. The Power and Size of the Tests

The power is defined as the probability to reject H_0 *under* $H_a : \theta = \theta_a$. It is written as $\pi(\theta_a) = P(reject\, H_0 \,|\, \theta = \theta_a)$. Similarly, the size is written as probability to reject H_0 *under* $H_0 : \theta = \theta_0$. To illustrate the graph of the power, we present the graphs of the power in testing parameters shape (δ, β) for one-side hypothesis on Weibull distribution, in Figure 2.

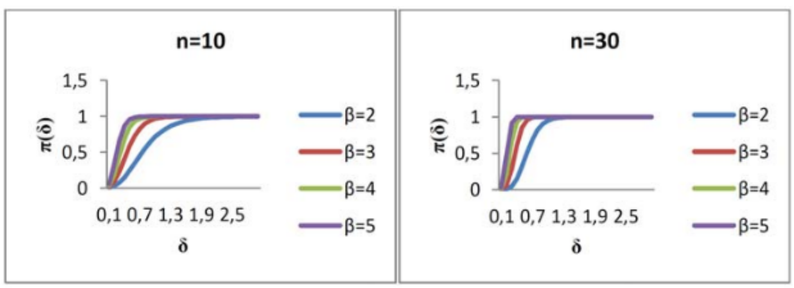

Figure 2. The graphs of power in testing parameter δ at $\alpha = 0, 1$.

It is clear (see Figure 2) that the power of the test increases as the sample size (n) and parameter shape β increase.

Many authors, such as Pratikno [3], Khan [20], Khan and Hoque [22], Yunus [16], Yunus and Khan [15], Khan and Pratikno [19], Khan et al. [26]

and Khan and Saleh [23-25] have used the power and size of the tests in testing intercept using non-sample prior information (NSPI). Following Pratikno [3], there are three tests for those, namely unrestricted test (UT), restricted test (RT) and pre-test test (PTT). In this case, the bivariate noncentral *F* distribution is used to compute the power of the pre-test test (PTT) on the multivariate simple regression model (MSRM). The formula of the power and size of the tests of the UT, RT and PTT are found in Pratikno [3] in testing hypothesis one-side or two-side hypothesis in MSRM. Here, the probability integral of the power and size of the PTT is very complex and not simple, so they are computed by R using bivariate noncentral *F* (BNCF) distribution. In this *R-code*, we modified the integral probability of the BNCF distribution using the square of the NCT distribution. A simulation is given using generated data from R in MSRM case. In this simulated example, we generated random data using R package. The explanatory variable (x) is generated from the uniform distribution between 0 and 1. The error vector ($\boldsymbol{e}$) is generated from the multivariate normal distribution with $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma = \sigma^2 \boldsymbol{I}_2$. Then, the dependent variable (y_i) is determined for fixed parameters β random. For the computation of the power functions of the tests we set $\alpha_1 = \alpha_2 = 0.05$, and generate $n = 30$ variates. The graphs of the power and size of the tests are produced using the formulas found in Pratikno [3]. The simulation graphs of the power and size of the tests for correlation coefficient 0.9 are then presented in Figure 2 and Figure 3 below.

From Figures 3 and 4, we have to choose the maximum power and minimum size to obtain the significant test. It is clear that the power of the PTT lies between UT and RT, and it increases as the UT and RT increase. Here, we see that the size of the PTT is the smallest (Figure 3), so the PTT would be an eligible choice of the test to be recommended.

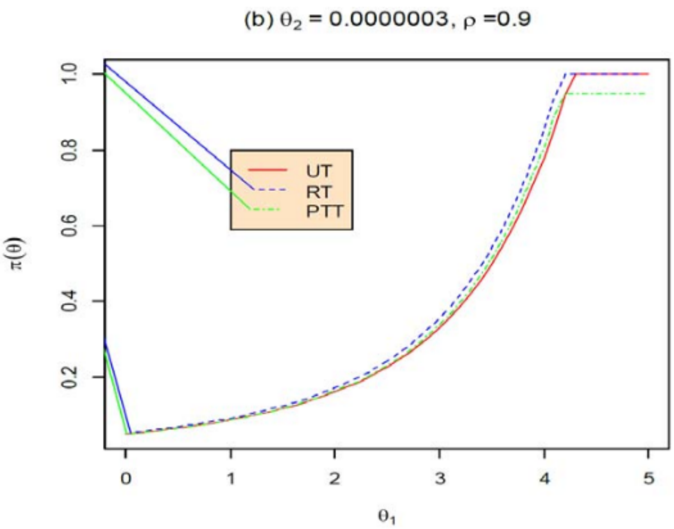

Figure 3. The power of the tests UT, RT and PTT.

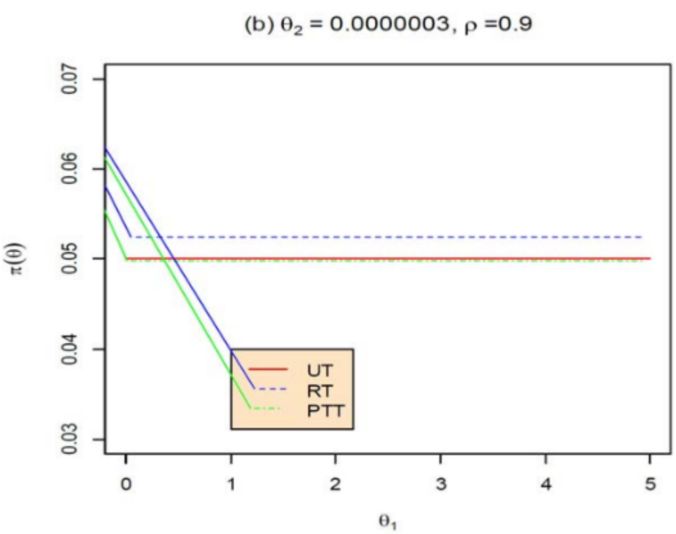

Figure 4. The size of the tests UT, RT and PTT.

4. Conclusion

We studied NCT distribution and its graphical analysis along with applications in the MSRM. To compute the power of the tests and plots of their graphs, *R-code* and square of the NCT distribution (as modified for BNCF) are used. The results showed that the curves of the NCT distribution tend to be skew positive for a large noncentral parameter. The sensitivity noncentral parameter occurred for $\delta \geq 1$. In terms of the power and size of the tests, the PTT could be a best choice of the test between UT and RT. This is due to the fact that the power of the PTT lies between UT and RT and the size of the PTT is minimum.

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