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by Budi Pratikno

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THE NONCENTRAL *F* DISTRIBUTION

B. Pratikno, Jajang, A. J. Putri and R. K. Aji

Department of Mathematics and Natural Science
Jenderal Soedirman University
Purwokerto, Indonesia

Abstract

This paper studies the probability distribution function (pdf), cumulative distribution function (cdf) and graphical analysis of a noncentral F distribution. The pdf formula of the noncentral F distributions is analytically derived, and the graphs are produced using R code. The results showed that: (1) the curves of the univarite noncentral F distribution decrease as the noncentral parameter increases, and they tend to be symmetric for large noncentrality parameter (≥ 54), and (2) the curves of the bivariate noncentral F (BNCF) increase as the degrees of freedom, coefficient correlation, and noncentrality parameter increase. For coefficient correlation (± 0.5), the graphs of the cdf of the *singly* BNCF distribution are identical.

1. Introduction

Many authors have already studied bivariate central F and univariate noncentral F distributions such as Krishnaiah [13, 14], Amos and Bulgren [4], Schuurmann et al. [6], El-Bassiouny and Jones [1], and Mudholkar et al. [8], Graybill [7], Muirhead [16] and Johnson et al. [11], respectively. One of

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them, Graybill [7] presented the pdf formula of the univariate noncentral F distribution. A lot of discussions on the approximation of the univariate noncentral F distribution are found in Mudholkar et al. [8] and Tiku [10]. Then, Tiku [10] and Krishnaiah and Armitage [15] discussed about multivariate noncentral F distribution. Some authors have also studied the univariate noncentral F distribution, such as Johnson et al. [11] and Shao [9]. Johnson et al. [11] provided the definition of the univariate noncentral F distribution known as the *singly and doubly* noncentral F distributions. Later, Yunus and Khan [17] derived the bivariate noncentral chi-square (BNCC) distribution by compounding the Poisson distribution with correlated bivariate central chi-square distribution. They also used noncentral F distribution to compute the power of the test in testing hypothesis with non-sample prior information (NSPI).

Saleh and Sen [2] and Yunus and Khan [17] have already used the cumulative distribution function (cdf) of a bivariate noncentral chi-square distribution to compute the power function of the test in a statistical test on the estimator parameters using NSPI. For large sample, the cdf of the bivariate noncentral chi-square (BNCC) distribution is used to compute the power function on multivariate simple regression model (MSRM). Similarly, in testing hypothesis with NSPI, the power and size of the tests are computed using BNCF (the bivariate noncentral F) distribution. This distribution is defined by mixing the correlated BNCC distribution with an independent central chi-square distribution, as well as compounding bivariate central F (BCF) with Poisson distributions. However, the probability distribution function (pdf) and cdf formula of the BNCF are very complicated. Here, Pratikno [3] already used the cdf of the BNCF to compute the power of the pre-test test (PTT) in testing intercept using NSPI on some regression models. Due to its complicated, then the pdf and cdf are computed using R code (R statistical package).

To describe the BNCF distribution in detail, we refer to Johnson et al. [11, p. 433, 435], that is, if $U_1, U_2, ..., U_v$ are independent normal variables with mean μ and variance σ^2, and $\partial_1, \partial_2, ..., \partial_v$ are nonzero constants, then the distribution of $\sum_{j=1}^{v}(U_j + \partial)^2$ follows a noncentral χ_v^2 distribution with v degrees of freedom (d.f.) and noncentrality parameter $\lambda = \sum_{j=1}^{v}\partial^2$. The cdf of noncentral chi-square distribution is then given in Patnaik [12]. Furthermore, Johnson et al. [11, p. 480] described the *doubly* noncentral F distribution with (v_1, v_2) degrees of freedom and noncentrality parameters λ_1 and λ_2 as the ratio between two independent noncentral chi-square variables, $\frac{\chi_{v_1}^2(\lambda_1)}{v_1}$ and $\frac{\chi_{v_2}^2(\lambda_2)}{v_2}$. The pdf and cdf of the singly and doubly correlated BNCF distributions are also found in Pratikno [3] and Khan et al. [19]. Here, the doubly correlated BNCF is defined by mixing the correlated BNCC distribution with an independent central chi-square distribution. This definition allows for two noncentrality parameters from the two correlated noncentral chi-square variables in the numerator of the noncentral F variables.

This paper studied the univariate noncentral F distribution in Section 2. The pdf and cdf of the BNCF distribution are derived in Section 3. The graphical analysis of the pdf and cdf of the BNCF distribution for some selected values of degrees of freedom and the noncentrality parameter are given in Section 4. Concluding remarks are provided in Section 5.

2. The Univariate Noncentral F Distribution

If w_1 and w_2 are two independent χ^2 random variables with n_1 and n_2 degrees of freedom, respectively, then

$$u = \frac{w_1/n_1}{w_2/n_2} \tag{1}$$

follows a central F_{n_1, n_2} distribution (Wackerly et al. [5, p. 362]). The pdf of the random variable u is then given by

$$f(u)=\frac{n_1^{n_1/2}n_2^{n_2/2}\Gamma\left(\frac{n_1+n_2}{2}\right)u^{(n_1-2)/2}}{\Gamma(n_1/2)\Gamma(n_2/2)(n_2+n_1u)^{(n_1+n_2)/2}},\quad u>0. \tag{2}$$

The mean and variance of this distribution are $\frac{n_2}{n_2-2}$ when $n_2>2$ and $\frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$ when $n_2>4$, respectively.

Following the concept of the central F distribution above and Graybill [7, p. 128], the noncentral F distribution is derived as follows. Let $w=\frac{x_1/n_1}{x_2/n_2}$, where x_1 is chi-square random variables with n_1 d.f. and noncentral parameter $(\lambda>0)$, and x_2 is chi-square random variables with n_2 d.f. and $\lambda=0$. The joint distribution of x_1 and x_2 is then given as

$$\begin{aligned}f(x_1,x_2)&=f(x_1).f(x_2)\\&=\sum_{j=0}^{\infty}\left(\frac{e^{-\lambda}\lambda^j}{j!}\right)\left(\frac{x_1^{\frac{(n_1+2j-2)}{2}}e^{-\frac{x_1}{2}}}{\Gamma\left(\frac{n_1+2j}{2}\right)2^{j+\frac{n_1}{2}}}\right).\frac{1}{\Gamma\left(\frac{n_2}{2}\right)2^{\frac{n_2}{2}}}x_2^{\left(\frac{n_2}{2}-1\right)}e^{-\frac{x_2}{2}}\\&=\sum_{j=0}^{\infty}\left(\frac{e^{-\lambda-\frac{x_1}{2}-\frac{x_2}{2}}\lambda^j x_1^{\frac{(n_1+2j-2)}{2}}x_2^{\left(\frac{n_2}{2}-1\right)}}{j!\,\Gamma\left(\frac{n_1+2j}{2}\right)\Gamma\left(\frac{n_2}{2}\right)2^{j+\frac{n_1}{2}+\frac{n_2}{2}}}\right).\end{aligned} \tag{3}$$

Let $z=x_1$ and $w=\frac{x_1/n_1}{x_2/n_2}=\frac{x_1n_2}{x_2n_1}$. Then the jacobian transformation is given as $J=\begin{vmatrix}\frac{\partial x_1}{\partial w} & \frac{\partial x_1}{\partial z}\\ \frac{\partial x_2}{\partial w} & \frac{\partial x_2}{\partial z}\end{vmatrix}=\left(\frac{z}{w^2}\right)\left(\frac{n_2}{n_1}\right)$. Furthermore, the joint distribution of z and w is written as

$$f(z, w) = f\left(z, \frac{zn_2}{wn_1}\right)|J|$$

$$= \sum_{j=0}^{\infty}\left(\frac{e^{-\lambda-\frac{z}{2}-\frac{zn_2}{2wn_1}}\lambda^j z^{\frac{(n_1+2j-2)}{2}}\left(\frac{zn_2}{wn_1}\right)^{\frac{n_2}{2}-1}}{j!\Gamma\left(\frac{n_1+2j}{2}\right)\Gamma\left(\frac{n_2}{2}\right)2^{j+\frac{n_1}{2}+\frac{n_2}{2}}}\right)\cdot\left(\frac{z}{w^2}\right)\left(\frac{n_2}{n_1}\right)$$

$$= \sum_{j=0}^{\infty}\left(\frac{e^{-\lambda}\lambda^j\left(\frac{n_2}{n_1}\right)^{\frac{n_2}{2}}}{j!\Gamma\left(\frac{n_1+2j}{2}\right)\Gamma\left(\frac{n_2}{2}\right)2^{j+\frac{n_1}{2}+\frac{n_2}{2}}}\right)$$

$$\times\left(\frac{z}{w}\right)^{\frac{(n_2-2)}{2}} z^{\frac{(n_1+2j-2)}{2}}\left(\frac{z}{w^2}\right)e^{-\frac{zn_2}{2wn_1}-\frac{z}{2}}.$$

So, the pdf of random variable w is obtained as

$$f(w) = \int_0^{\infty} f(z, w)dz, \tag{4}$$

that is,

$$f(w) = \int_0^{\infty}\left[\sum_{j=0}^{\infty}\frac{e^{-\lambda}\lambda^j\left(\frac{n_2}{n_1}\right)^{\frac{n_2}{2}}}{j!\Gamma\left(\frac{n_1+2j}{2}\right)\Gamma\left(\frac{n_2}{2}\right)2^{j+\frac{n_1}{2}+\frac{n_2}{2}}}\right.$$

$$\left.\times\left(\frac{z}{w}\right)^{\frac{(n_2-2)}{2}} z^{\frac{(n_1+2j-2)}{2}}\left(\frac{z}{w^2}\right)e^{-\frac{zn_2}{2wn_1}-\frac{z}{2}}\right]dz$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^{j}\left(\frac{n_1}{n_2}\right)^{\frac{n_1+2j}{2}} w^{\frac{n_1+2j-2}{2}}}{j!\Gamma\left(\frac{n_1+2j}{2}\right)\Gamma\left(\frac{n_2}{2}\right)2^{j+\frac{n_1}{2}+\frac{n_2}{2}}\left(1+\frac{wn_1}{n_2}\right)^{\frac{n_1+n_2+2j}{2}}}$$

$$\times \int_0^{\infty} y^{\frac{n_1}{2}+\frac{n_2}{2}+j-1} e^{-\frac{1}{2}y} dy.$$

Finally, we get the pdf of the univariate noncentral F distribution as

$$f(w) = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^{j}\left(\frac{n_1}{n_2}\right)^{\frac{n_1+2j}{2}} w^{\frac{n_1+2j-2}{2}}\Gamma\left(j+\frac{n_1}{2}+\frac{n_2}{2}\right)}{j!\Gamma\left(\frac{n_1+2j}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\left(1+\frac{wn_1}{n_2}\right)^{\frac{n_1+n_2+2j}{2}}}. \tag{5}$$

Following equation (5), for $n_1 = n_2 = 60$ and $\lambda = 1, 3, 9, 18, 27, 35, 54, 100$, the curves of the pdf univariate noncentral F distribution are then presented in Figure 1.

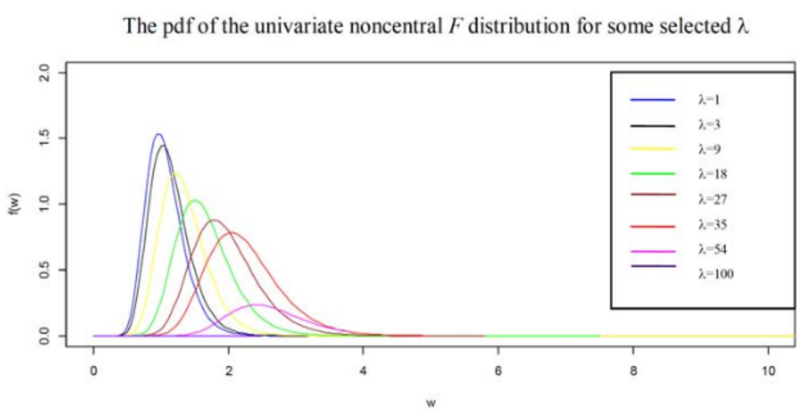

Figure 1. The pdf curve of univariate noncentral F distribution for $n_1 = n_2 = 60$ and some selected λ.

For $\lambda = 0$, the curve will be the pdf central F distribution. From a simulation, for $\lambda < 1$, we obtain that all the curves are *skew positive*. Figure 1 shows that the peak points of the curves of the noncentral F distribution decrease (and the tail increases) as the noncentral parameters increase, and they will be then symmetric for large λ $(\lambda = 54)$. Moreover, they will be linear for $\lambda \to \infty$.

3. The Bivariate Noncentral F Distribution

Following Pratikno [3] and Khan et al. [19], the pdf and cdf of the bivariate noncentral F (BNCF) distribution are derived as follows. Following Krishnaiah [13] for $x_i = F_i \sim F_{\nu_1, \nu_2}$ with $i = 1, 2,$ (ν_1, ν_2) degrees of freedom and correlation coefficient (ρ), the pdf and cdf of the BCF distribution are defined, respectively, as

$$f(x_1, x_2) = \left(\frac{\nu_2^{\nu_2/2}(1-\rho^2)^{(\nu_1+\nu_2)/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\right)$$
$$\times \sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_1 + (\nu_2/2) + 2j)}{j!\,\Gamma(\nu_1/2) + j}\right)\nu_1^{\nu_1+2j}$$
$$\times \left(\frac{(x_1x_2)^{(\nu_1/2)+j-1}}{[\nu_2(1-\rho^2) + \nu_1(x_1 + x_2)]^{\nu_1+(\nu_2/2)+2j}}\right), \tag{6}$$

and

$$P(x_1 < d, x_2 < d) = \left(\frac{(1-\rho^2)^{\nu_1/2}}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}\right)$$
$$\times \sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(\nu_1 + (\nu_2/2) + 2j)}{j!\,\Gamma(\nu_1/2) + j}\right)L_j, \tag{7}$$

where $L_j = \int_0^h\int_0^h \frac{(x_1x_2)^{(\nu_1/2)+j-1}dx_1dx_2}{(1 + x_1 + x_2)^{\nu_1+(\nu_2/2)+2j}}$, with $h = \frac{d\nu_1}{\nu_2(1-\rho^2)}$. An approximation to the value of the cdf of the BCF distribution is also found in Amos and Bulgren [4]. Furthermore, Johnson et al. [11, p. 480] described the *doubly* noncentral F variable with (v_1, v_2) degrees of freedom and noncentrality parameters λ_1 and λ_2 is defined as $F''_{v_1, v_2}(\lambda_1, \lambda_2) = \dfrac{\chi'^2_{v_1}(\lambda_1)/v_1}{\chi'^2_{v_2}(\lambda_2)/v_2}$, where the two noncentral chi-square distributions are independent. In many applications $\lambda_2 = 0$, arises when there is a central χ^2 variable in the denominator of F''_{v_1, v_2}. This is called a *singly* noncentral (or simply noncentral) F variable with (v_1, v_2) degrees of freedom and noncentrality parameter λ_1. The case for $\lambda_1 = 0$, $\lambda_2 \neq 0$ is not considered here, but note that $F''_{v_1, v_2}(0, \lambda_2) = \dfrac{1}{F'_{v_1, v_2}(\lambda_2)}$, $E[F'_{v_1, v_2}(\lambda_1)] = \dfrac{v_2(v_1 + \lambda_1)}{v_1(v_2 - 2)}$, $v_2 > 2$, and

$$Var[F'_{v_1, v_2}(\lambda_1)] = 2\left(\frac{v_2}{v_1}\right)^2\left(\frac{(v_1 + \lambda_1)^2 + (v_1 + 2\lambda_1)(v_2 - 2)}{(v_2 - 4)(v_2 - 2)^2}\right), \quad v_2 > 4.$$

Moreover, Johnson et al. [11, p. 499] described

$$G''_{v_1, v_2}(\lambda_1, \lambda_2) = \frac{\chi'^2_{v_1}(\lambda_1)}{\chi'^2_{v_2}(\lambda_2)}$$

as a mixture of G_{v_1+2j, v_2+2k} distribution in proportion of $\left(\dfrac{e^{-\lambda_1/2}\left(\frac{\lambda_1}{2}\right)^j}{j!}\right) \times \left(\dfrac{e^{-\lambda_2/2}\left(\frac{\lambda_2}{2}\right)^j}{k!}\right)$, representing product of two independent Poisson distributions. The details of the pdf and cdf of G'' are also found in Johnson et al. [11, p. 500].

Later, following Krishnaiah [13] and Johnson et al. [11, p. 499], for $x_i = F_i \sim F_{v_1, v_2}$ with $i = 1, 2,$ and $r \sim Pois(\lambda)$ the *singly* BNCF variable is given as $\sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right) \times F_{v_r, v_2}$, where $v_r = v_1 + 2r$. Furthermore, the pdf of the *singly* BNCF distribution with noncentrality parameter λ is given by the pdf

$$f(x_1, x_2, v_r, v_2, \lambda) = \sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right) f_1(x_1, x_2, v_r, v_2), \qquad (8)$$

where $f_1(x_1, x_2, v_r, v_2)$ is the pdf of a BCF distribution with v_r and v_2 d.f., that is,

$$f_1(x_1, x_2, v_r, v_2) = \left(\frac{v_2^{v_2/2}(1-\rho^2)^{(v_r+v_2)/2}}{\Gamma(v_r/2)\Gamma(v_2/2)}\right) \times \sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(v_r + (v_2/2) + 2j)}{j!\,\Gamma((v_r/2) + j)}\right) \times (v_r^{v_r+2j})\left(\frac{(x_1 x_2)^{(v_r/2)+j-1}}{[v_2(1-\rho^2) + v_r(x_1 + x_2)]^{v_r+(v_2/2)+2j}}\right). \qquad (9)$$

Then the cdf of the *singly* BNCF distribution is defined as

$$P(.) = P(x_1 < d, x_2 < d, v_r, v_2, \lambda) = \sum_{r=0}^{\infty}\left(\frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^r}{r!}\right) \times P_2(x_1 < d, x_2 < d, v_r, v_2), \qquad (10)$$

where

$$P_2(x_1 < d,\, x_2 < d,\, v_r,\, v_2) = \left(\frac{(1-\rho^2)^{v_r/2}}{\Gamma(v_r/2)\Gamma(v_2/2)}\right) \times \sum_{j=0}^{\infty}\left(\frac{\rho^{2j}\Gamma(v_r + (v_2/2) + 2j)}{j!\,\Gamma((v_r/2)+j)}\right)L_{jr} \quad (11)$$

in which L_{jr} is defined as $L_{jr} = \int_0^h\int_0^h \frac{(x_1x_2)^{(v_1/2)+j-1}dx_1dx_2}{(1+x_1+x_2)^{v_r+(v_2/2)+2j}}$ with $h_r = \frac{dv_r}{v_2(1-\rho^2)}$. For the computation of the value of the cdf of the *singly* BNCF distribution, R code is used.

Furthermore, we derive the *doubly* BNCF distribution that is defined by compounding the pdf of the BNCC distribution of x_1 and x_2 with m degrees of freedom and noncentrality parameters θ_1 and θ_2 and central chi-square distribution of z with n degrees of freedom (see Yunus and Khan [18], and Amos and Bulgren [4]). The pdf of the BNCC variables x_1 and x_2 is given by

$$g(x_1, x_2) = \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}[\rho^{2j}(1-\rho^2)^{m/2}\Gamma(m/2+j)] \times \left[\frac{(x_1)^{m/2+j+r_1-1}e^{-\frac{(x_1)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)} \times \frac{e^{-\theta/2}(\theta/2)^{r_1}}{r_1!}\right] \times \left[\frac{(x_2)^{m/2+j+r_2-1}e^{-\frac{(x_2)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)} \times \frac{e^{-\theta/2}(\theta/2)^{r_2}}{r_2!}\right], \quad (12)$$

and the pdf of a central chi-square variable z with n degrees of freedom is

$$f(z) = \frac{z^{(n/2)-1} e^{-z/2}}{2^{n/2}\Gamma(n/2)}, \quad z > 0, \tag{13}$$

where z is independent of x_1 and x_2. Then using transformation of variable method for multivariable case (see for instance, Wackerly et al. [5, p. 325]), we obtain the joint pdf of $y = [y_1, y_2]'$ and z variables as

$$f(y, z) = f(x) f(z) | J((x, z) \to (y, z)) |, \tag{14}$$

where $y_1 = \frac{nx_1}{mz}$, $y_2 = \frac{nx_2}{mz}$ and the jacobian transformation $(x_1, x_2, z) \to (y_1, y_2, z)$, is given by $\left(\frac{m}{n} z\right)^2$.

Therefore, the joint pdf of y and z is given by

$$\begin{aligned} f(y, z) = {} & \sum_{j=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} [\rho^{2j} (1 - \rho^2)^{m/2} \Gamma(m/2 + j)] \\ & \times \left[\frac{\left(\frac{m}{n} y_1 z\right)^{m/2+j+r_1-1} e^{-\frac{\left(\frac{m}{n} y_1 z\right)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_1} \Gamma(m/2 + j + r_1)} \times \frac{e^{-\theta_1/2} (\theta_1/2)^{r_1}}{r_1!} \right] \\ & \times \left[\frac{\left(\frac{m}{n} y_2 z\right)^{m/2+j+r_2-1} e^{-\frac{\left(\frac{m}{n} y_2 z\right)}{2(1-\rho^2)}}}{[2(1-\rho^2)]^{m/2+j+r_2} \Gamma(m/2 + j + r_2)} \times \frac{e^{-\theta_2/2} (\theta_2/2)^{r_2}}{r_2!} \right] \\ & \times \frac{z^{(n/2)-1} e^{-z/2}}{2^{n/2}\Gamma(n/2)} \times \left(\frac{m}{n} z\right)^2. \end{aligned} \tag{15}$$

Then the joint pdf of (y_1, y_2), where

$$y_i = \frac{x_i/m}{z/n}, \text{ for } i = 1, 2, \tag{16}$$

is obtained as $f(y) = f(y_1, y_2) = \int_z f(y, z)dz$, that is

$$f(y_1, y_2) = \left(\frac{m}{n}\right)^2 \left[\frac{(1-\rho^2)^{m/2}}{2^{n/2}\Gamma(n/2)}\right] \times \sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}[\rho^{2j}\Gamma(m/2+j)]$$

$$\times \left(\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right)\left(\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right)$$

$$\times \left[\frac{\left(\frac{m}{n}y_1 z\right)^{m/2+j+r_1-1}}{[2(1-\rho^2)]^{m/2+j+r_1}\Gamma(m/2+j+r_1)}\right]$$

$$\times \left[\frac{\left(\frac{m}{n}y_2 z\right)^{m/2+j+r_2-1}}{[2(1-\rho^2)]^{m/2+j+r_2}\Gamma(m/2+j+r_2)}\right]$$

$$\times \int_0^{\infty} z^{m+n/2+2j+r_1+r_2-1} e^{-z/2\left(\frac{(m/n)y_1}{1-\rho^2}+\frac{(m/n)y_2}{1-\rho^2}+1\right)}dz$$

$$= \left(\frac{m}{n}\right)^m \left[\frac{(1-\rho^2)^{\frac{m+n}{2}}}{\Gamma(n/2)}\right]\sum_{j=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\left[\rho^{2j}\left(\frac{m}{n}\right)^{2j}\Gamma(m/2+j)\right]$$

$$\times \left[\left(\frac{e^{-\theta_1/2}(\theta_1/2)^{r_1}}{r_1!}\right)\left(\frac{\left(\frac{m}{n}\right)^{r_1}}{\Gamma(m/2+j+r_1)}\right)(y_1^{m/2+j+r_1-1})\right]$$

$$\times\left[\left(\frac{e^{-\theta_2/2}(\theta_2/2)^{r_2}}{r_2!}\right)\left(\frac{\left(\frac{m}{n}\right)^{r_2}}{\Gamma(m/2+j+r_2)}\right)(y_2^{m/2+j+r_2-1})\right]$$

$$\times\,\Gamma(q_{rj})\left[(1-\rho^2)+\frac{m}{n}y_1+\frac{m}{n}y_2\right]^{-(q_{rj})}, \tag{17}$$

where $w_y = \frac{(m/n)y_1}{1-\rho^2} + \frac{(m/n)y_2}{1-\rho^2} + 1$ and $q_{rj} = m + n/2 + 2j + r_1 + r_2$. The cdf of the doubly BNCF distribution is then defined as

$$P(y_1 < a,\ y_2 < b) = \int_0^a\int_0^b f(y_1,\ y_2)dy_1dy_2, \tag{18}$$

where a and b are positive real numbers. The above cdf can be expressed as

$$P(y_1 < a,\ y_2 < b) = \int_0^\infty f(z)\int_0^{\frac{bmy}{n}}\int_0^{\frac{amy}{n}} g(x_1,\ x_2)dx_1dx_2dz, \tag{19}$$

where $g(x_1, x_2)$ is the pdf of the BNCC distribution which is given in equation (12), $f(z)$ is the pdf of the central chi-square variable which is given in equation (13), and y_i for $i = 1, 2,$ are given in equation (16). Finally, we note that equation (19) is called the *cdf of BNCF distribution*.

4. Graphical Analysis of the cdf of the BNCF Distribution

Following Pratikno [3] and Khan et al. [19], the graph of the cdf of the *singly* BNCF and *doubly* distributions are presented in Figure 2, which shows that the value of cdf of the *singly* BNCF distributions increases as the value of any of the parameters, degrees of freedom v_1 (for fixed v_2), λ and d, increases. From equation (8), we see that the cdf of the *singly* BNCF distribution also depends on ρ. For both $\rho < 0.5$ and $\rho > 0.5$, the curves of the cdf of the *singly* BNCF distribution are lower than that for $\rho = 0.5$. Note that (in simulation), for $\rho = -0.5$ and $\rho = 0.5$, the graphs of the cdf of the *singly* BNCF distribution are identical. The graphs of the cdf of the BNCF are not presented, but the values of the cdf of the BNCF distribution are given in Table 1. Details of the curves of the cdf of the BNCF are found in Khan et al. [19] and Pratikno [3]. Table 1 shows the values of the cdf of the BNCF distribution approaches 1 quicker for large correlation (ρ) and d, and these increase as the ρ and d increase. It means that the curve of the cdf of the BNCF will also increase as the ρ and d increase.

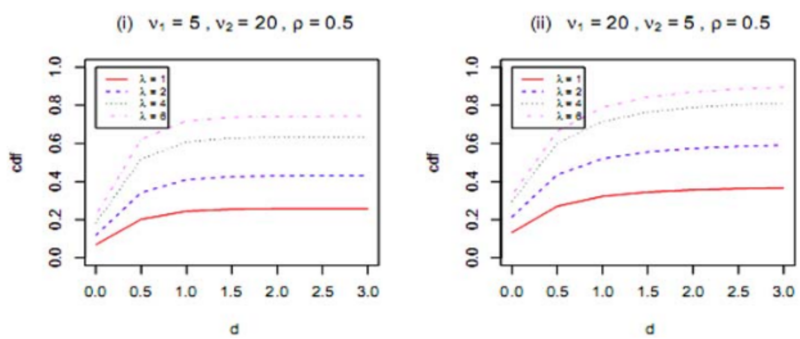

Figure 2. The curve of the cdf of singly bivariate noncentral F distribution.

Table 1. The values of the cdf of the BNCF distribution for $m = 10$, $n = 20$, $\theta_1 = 1$, $\theta_2 = 1, 5$; $d = 0, 5; 1; 1, 5; 2; 2, 5; 3$ and $\rho = -0, 5; -0, 3; 0; 0, 3; 0, 5$

d	ρ				
	–0, 5	–0, 3	0	0, 3	0, 5
0, 5	0, 02413	0, 01868	0, 01627	0, 01868	0, 02413
1	0, 28612	0, 25805	0, 24381	0, 25805	0, 28612
1, 5	0, 60971	0, 58102	0, 56546	0, 58102	0, 60971
2	0, 80984	0, 79072	0, 77986	0, 79072	0, 80984
2, 5	0, 90967	0, 89863	0, 89212	0, 89863	0, 90967
3	0, 95657	0, 95049	0, 94678	0, 95049	0, 95657

Furthermore, Pratikno [3] used the cdf of the bivariate noncentral *F* (BNCF) distribution to compute and figure the power of the unrestricted test (UT), restricted test (RT) and pre-test test (PTT) on *bivariate simple regression model* (BSRM) (see Figure 3), that is, the linear relationship between two responses (vector) and a single predictor (x_i, y_i), for $i = 1, 2, ..., n$, is modeled as $\boldsymbol{y}_i = \beta_0 + \beta_1 x_i + e_i$. Here, $\boldsymbol{y}_i = (y_{i1}, ..., y_{ip})'$ is the p dimensional ith response vector, x_i is the ith nonzero scalar value of the explanatory variable, $\boldsymbol{\beta_0} = (\beta_{01}, ..., \beta_{0p})'$ and $\boldsymbol{\beta_1} = (\beta_{11}, ..., \beta_{1p})'$ are $p \times 1$ column vectors of unknown intercept and slope parameters, respectively, and $\mathbf{e_i} = (e_{i1}, ..., e_{ip})'$ is the $p \times 1$ dimensional ith vector of errors with $\mathbf{e_i} \sim Np(0, \Sigma)$. Later, the integral probability of the cdf of the BNCF of the PTT on the BSRM is computed using R code. Detailed formula of the power of the tests of the UT, RT and PTT are found at Pratikno [3].

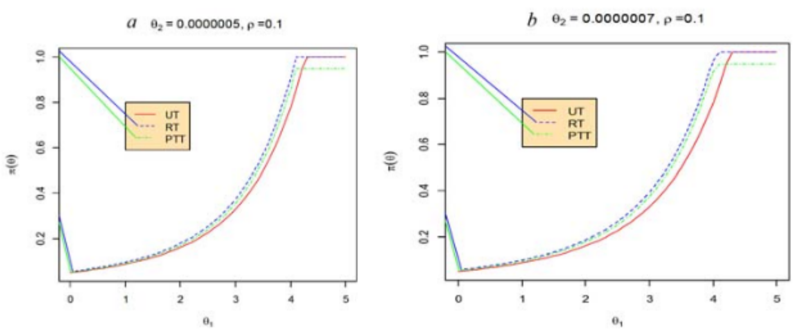

Figure 3. The power of the tests of the UT, RT and PTT at $\rho = 0.1$.

From Figure 3, we see that the curves are sigmoid and the PTT lies between RT and UT. In the simulation for some selected coefficient correlation ($\rho = \pm 0.3$; 0.5), the graphs are similar, and these increase as the ρ and d increase. Following Pratikno [3] and previous researches, the PTT will be an alternative choice between them.

5. Concluding Remark

The curve of the univarite noncentral *F* distribution declines (and/or the tail increases) as the noncentral parameter increases. For large noncentrality parameter λ $(\lambda = 54)$, the curve tends to be symmetric. On the other hand, the curves of the bivariate noncentral *F* (BNCF) distribution are depended on degrees of freedom, coefficient correlation and noncentrality parameter. These (the cdf of the *singly* and *doubly*) BNCF distributions increase as the value of any of the parameters, degrees of freedom v_1 (for fixed v_2), noncentrality parameter, increase. For $\rho = -0.5$ and $\rho = 0.5$, the graphs of the cdf of the *singly* BNCF distribution are identical. The cdf of the BNCF distribution approaches 1 quicker for large correlation (ρ).

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