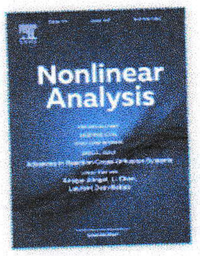

ISSN: 0362-546X

Submit Your Paper

View Articles

Guide for Authors

Abstracting/ Indexing

Track Your Paper

Order Journal

Journal Metrics

CiteScore: 1.30

Nonlinear Analysis

An International Mathematical Journal

> Supports Open Access

Editors-in-Chief: Siegfried Carl, Enzo Luigi Mitidieri

> View Editorial Board

For more information on our journals visit: http://www.elsevier.com/mathematics

Nonlinear Analysis focuses on papers that address significant problems in Nonlinear Analysis that have a sustainable and important impact on the development of new directions in the theory as well as potential applications. Review articles on important topics in Nonlinear Analysis are welcome as well...

Read more

Most Downloaded | Recent Articles | Most Cited | Open Access Articles

Basic theory of fractional differential equations V. Lakshmikantham | A.S. Vatsala

Strongly stable surfaces in sub-Riemannian 3-space forms Ana Hurtado | César Rosales

A priori bounds for superlinear elliptic equations with semidefinite nonlinearity Yūki Naito | Takashi Suzuki | ...

View All Articles >

Journal Metrics

CiteScore: **1.30**

More about CiteScore

Impact Factor: **1.125**

5-Year Impact Factor: **1.357**

Source Normalized Impact per Paper (SNIP): **1.407**

SCImago Journal Rank (SJR): **1.481**

> View More on Journal Insights

Article Enrichments

> AudioSlides

> Interactive MATLAB Figure Viewer

> Interactive Plot Viewer

Related Links

> Author Stats

> Publishing Campus

Forthcoming Special Issues

Contributions to special issues are requested by personalized invitation only

Nonlinear PDEs and Geometric Function Theory, in honor of Carlo Sbordone on his 70th birthday

Advances in Reaction-Cross-Diffusion Systems

Contributions to special issues are requested by personalized invitation only

> View All

News

Honoring Srinivasa Ramanujan on Pi Day

Release of STIX Fonts Version 2.0.0

Apply now- Elsevier Mathematical Sciences Sponsorship Fund

> View All

MSN Japan - Hotmail, Outl… | nonlinear analysis: theory m… | Nonlinear Analysis | Vol 141

www.sciencedirect.com/science/journal/0362546X/141?sdc=1

Nonlinear Analysis

Supports Open Access | About this Journal | Sample Issue Online | Submit your Article

Get new article feed

Get new Open Access article feed

Subscribe to new volume alerts

Add to Favorites

Nonlinear Analysis
Volume 141, Pages 1-210 (August 2016)

Articles 1 - 11

Purchase | Download PDFs | Export | All access types

Type here to search

4:14 PM
9/25/2017

Journal Metrics

CiteScore: 1.30

More about CiteScore

Impact Factor: 1.192

5-Year Impact Factor: 1.353

Source Normalized Impact per Paper (SNIP): 1.273

SCImago Journal Rank (SJR): 1.474

View More on Journal Insights

Nonlinear Analysis Editorial Board

Honorary Editors

Editorial Board

N. Garofalo

Università degli Studi di Padova, Padova, Italy
PDE's in Analysis and Geometry, Geometric flows and related PDE's, Sub-Riemannian geometry, Free Boundary problem

F. Gazzola

Politecnico di Milano, Milano, Italy
Partial Differential Equations, Calculus of Variations

G. Grillo

Politecnico di Milano, Milano, Italy
Nonlinear Evolution Equations, Singular and Degenerate Parabolic Equations, Functional Inequalities, Analysis on Manifolds.

P. Hästö

University of Turku, Turku, Finland
Theory of function spaces, Calculus of variation, partial differential equations with nonstandard growth

T. Kilpeläinen

University of Jyväskylä, Jyväskylä, Finland
Nonlinear partial differential equations, Measure data problems, Nonlinear potential theory

P. Marcati

Università degli Studi dell' Aquila, L'Aquila, Italy
Nonlinear Hyperbolic Systems, Compressible and Incompressible Fluids, Nonclassical fluids, evolution PDEs in quantum mechanics.

G. Mingione

Università degli Studi di Parma, Parma, Italy
Nonlinear partial differential equations, Calculus of Variations, Regularity theory

R. Monti

Università degli Studi di Padova, Padova, Italy
Analysis in Carnot-Carathéodory spaces

D. Motreanu

Université de Perpignan Via Domitia (UPVD), Perpignan, France
Nonlinear elliptic boundary value problems, Nonlinear eigenvalue problems, Variational methods

P. Omari

Università di Trieste, Trieste, Italy
Ordinary Differential Equations: Topological, Variational, and Monotone Methods

P. Pucci

Università degli Studi di Perugia, Perugia, Italy
Quasilinear elliptic equations and inequalities, problems with critical growth, p- Kirchhoff type equations

V. Radulescu

Romanian Academy, Bucharest, Romania
Degenerate and Singular Elliptic Equations, Nonlinear Operators, Global Analysis, Variational Inequalities

M. Růžička

Albert-Ludwigs-Universität Freiburg, Freiburg, Germany
Non-Newtonian fluids, Nonlinear elliptic and Parabolic PDEs

S. Salsa

Politecnico di Milano, Milano, Italy
Local and non local elliptic and parabolic equations, free boundary problems.

Ph. Souplet

Université Paris-Nord (Paris XIII), Villetaneuse, France
Nonlinear partial differential equations of parabolic and elliptic type

Nonlinear Analysis Editorial Board

L. D'Ambrosio

Università di Bari, Bari, Italy
Hardy inequalities, Liouville Theorems, subelliptic Nonlinear PDE's

F. Demengel

Université de Cergy-Pontoise, Cergy-Pontoise, France
NonLinear Pde's, One Laplacian, Viscosity solutions for singular or degenerate Fully Nonlinear equations, Regularity.

S. Dragomir

Università della Basilicata, Potenza, Italy
Nonlinear PDEs of variational origin on CR and Lorentzian manifolds, (subelliptic) harmonic maps, Yang-Mills fields

J. Escher

Leibniz Universität Hannover, Hannover, Germany
Free and Moving Boundary Problems, Qualitative Theory of Nonlinear Elliptic and Parabolic Problems, Geometrical Evolution Equations, Mathematical Fluid Dynamics

L. Fanelli

Università di Roma "La Sapienza", Roma, Italy
Dispersive PDE's, Scattering Theory, Fourier Analysis

A. Farina

Université de Picardie Jules Verne, Amiens CEDEX 1, France
Nonlinear Partial Differential Equations, Qualitative properties and classification of solutions, Geometric and Variational Methods, Analysis on manifolds

A. Friedman

The Ohio State University, Columbus, Ohio, USA
Partial Differential Equation, Mathematical Modeling of Physical and Biological Processes

N. Fusco

Università degli studi di Napoli Federico II, Napoli, Italy
Calculus of Variations, Elliptic and Parabolic Differential Equations, Geometric Measure Theory

T. Gallouët

Aix-Marseille Université, Marseille Cedex 13, France
Nonlinear Partial Differential Equations, Numerical Analysis

N. Garofalo

Purdue University, West Lafayette, IN, USA, Dept. of Structural and Transportation Engineering (DICEA), University of Padova, Italy
PDE's in Analysis and Geometry, Geometric flows and related PDE's, Sub-Riemannian geometry, Free Boundary problem

F. Gazzola

Nonlinear Analysis 141 (2016) 109–129

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

On the free boundary problem for the Oldroyd-B Model in the maximal L_p–L_q regularity class

Sri Maryani*

Department of Pure and Applied Mathematics, Graduate School of Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan

ARTICLE INFO

Article history:
Received 11 January 2016
Accepted 31 March 2016
Communicated by S. Carl

MSC:
35Q35
76N10

Keywords:
Non-Newtonian compressible viscous barotropic fluid flow
Oldroyd B type
Local well-posedness
Maximal L_p–L_q regularity

ABSTRACT

In the present work, we prove the local well-posedness of non-Newtonian compressible viscous barotropic fluid flow of Oldroyd-B type with free surface in a bounded domain of N-dimensional Euclidean space ($N \geq 2$). The key step is to prove the maximal L_p–L_q regularity theorem for the linearized equation with the help of the $\mathcal{R}$-bounded solution operators for the corresponding resolvent problem and Weis's operator valued Fourier multiplier theorem.

1. Introduction and main result

Let Ω be a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) whose boundary consists of two parts Γ_0 and Γ_1, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. The Ω is occupied by a compressible viscous barotropic non-Newtonian fluid of Oldroyd-B type. The present paper deals with the problem of determining the region $\Omega_t \subset \mathbb{R}^N$, the density field $\rho = \rho(x,t)$, the elastic tensor $\tau = \tau(x,t)$, and the velocity field $\mathbf{u} = (u_1(x,t), \ldots, u_N(x,t))$, which satisfy the system of equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega_t, \\ \rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{Div}\mathbf{T}(\mathbf{u}, P(\rho)) = \beta \operatorname{Div}\tau & \text{in } \Omega_t, \\ \partial_t \tau + \mathbf{u}\cdot\nabla\tau + \gamma\tau = \delta\mathbf{D}(\mathbf{u}) + g_\alpha(\nabla\mathbf{u}, \tau) & \text{in } \Omega_t, \\ (\mathbf{T}(\mathbf{u}, P(\rho)) + \beta\tau)\mathbf{n}_t = -P(\rho_*)\mathbf{n}_t & \text{on } \Gamma_t, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \\ (\rho, \mathbf{u}, \tau)|_{t=0} = (\rho_* + \theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega, \\ \Omega_t|_{t=0} = \Omega_0, \qquad \Gamma_t|_{t=0} = \Gamma_1 \end{cases} \tag{1.1}$$

* Correspondence to: Department of Mathematics, Jenderal Soedirman University, Indonesia. Tel.: +81 3 5286 3000.
E-mail address: sri.maryani@fuji.waseda.jp.

http://dx.doi.org/10.1016/j.na.2016.03.024
0362-546X/

for $0 < t < T$. Here, ρ_* is a positive constant describing the mass density of the reference domain Ω, $\mathbf{T}(\mathbf{u}, P(\rho))$ the stress tensor of the form

$$\mathbf{T}(\mathbf{u}, \rho) = \mathbf{S}(\mathbf{u}) - P(\rho)\mathbf{I} \quad \text{with } \mathbf{S}(\mathbf{u}) = \mu\mathbf{D}(\mathbf{u}) + (\nu - \mu)\mathrm{div}\,\mathbf{u}\mathbf{I}, \tag{1.2}$$

$\mathbf{D}(\mathbf{u})$ the doubled deformation tensor whose (i, j) components are $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$ ($\partial_i = \partial/\partial x_j$), $\mathbf{I}$ the $N \times N$ identity matrix, μ, ν, β, γ and δ are positive constants (μ and ν are the first and second viscosity coefficients, respectively), $\mathbf{n}_t$ is the unit outer normal to Γ_t, $P(\rho)$ a C^∞ function defined for $\rho > 0$ which satisfies that $P'(\rho) > 0$ for $\rho > 0$. Moreover, the function $g_\alpha(\nabla\mathbf{u}, \tau)$ has a form

$$g_\alpha(\nabla\mathbf{u}, \tau) = \mathbf{W}(\mathbf{u})\tau - \tau\mathbf{W}(\mathbf{u}) + \alpha(\tau\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\tau), \tag{1.3}$$

where α is a constant with $-1 \le \alpha \le 1$ and $\mathbf{W}(\mathbf{u})$ the doubled antisymmetric part of the gradient $\nabla\mathbf{u}$ whose (i, j) components are $W_{ij}(\mathbf{u}) = \partial_i u_j - \partial_j u_i$. Finally, for any matrix field $\mathbf{K}$ whose components are K_{ij}, the quantity $\mathrm{Div}\,\mathbf{K}$ is an N vector whose ith component is $\sum_{j=1}^N \partial_j K_{ij}$, and also for any vector of functions $\mathbf{u} = (u_1, \ldots, u_N)$, $\mathrm{div}\,\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, and $\mathbf{u} \cdot \nabla\mathbf{u}$ is an N vector whose ith component is $\sum_{j=1}^N u_j\partial_j u_i$. We assume that the boundary of Ω_t consists of Γ_0 and Γ_t with $\Gamma_0 \cap \Gamma_t = \emptyset$.

Aside from the dynamical system (1.1), a further kinematic condition for Γ_t is satisfied, which gives

$$\Gamma_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in \Gamma_1)\}, \tag{1.4}$$

where $\mathbf{x} = \mathbf{x}(\xi, t)$ is the solution to the Cauchy problem:

$$\Gamma_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in \Gamma_1)\}. \tag{1.5}$$

Concerning the free boundary problem of the viscous compressible barotropic Newtonian fluid flow, the local well-posedness and global well-posedness have been studied in the L_2 Sobolev–Slobodetskii space by Denisova and Solonnikov [4,3], Secchi and Valli [17–19], Solonnikov and Tani [28,30,31], and Zajaczkowski [34,35], and in the L_p–L_q maximal regularity class by Shibata et al. [7,24]. Recently, M. Nesensohn [14] proved the local well-posedness of the free boundary problem for the non-Newtonian fluid flow of Oldroyd-B type in the incompressible viscous fluid case (further references are found in [14]). On the other hand, Shi, Wang and Zhang [20] investigated the asymptotic stability for 1-dimensional motion of non-Newtonian compressible fluids using L_2 energy method. Meanwhile, global existence of strong solutions of Navier–Stokes equations with non-Newtonian potential for 1-dimensional isentropic compressible fluids has been studied by Liu, Yuan and Lie [9]. The purpose of this paper is to study the local well-posedness of problem (1.1).

To prove the local well-posedness of problem (1.1), we use the Lagrangian coordinate in order to transform the time dependent domain Ω_t to the fixed domain Ω. Let $\mathbf{u}(x, t)$ and $\mathbf{v}(\xi, t)$ be velocity fields in the Euler coordinate and in the Lagrangian coordinate, respectively. The Euler coordinate system $\{x\}$ and Lagrangian coordinate system $\{\xi\}$ are connected by the relation:

$$x = \xi + \int_0^t \mathbf{v}(\xi, s)ds \equiv \mathbf{X}_\mathbf{v}(\xi, t),$$

where, $\mathbf{v}(\xi, t) = (v_1(\xi, t), \ldots, v_N(\xi, t)) = \mathbf{u}(\mathbf{X}_\mathbf{v}(\xi, t), t)$. Let A be the Jacobi matrix of the transformation $x = \mathbf{X}_\mathbf{v}(\xi, t)$, whose (i, j) element is $a_{ij} = \delta_{ij} + \int_0^t (\frac{\partial v_i}{\partial \xi_j})(\xi, s)ds$. There exists a small number σ such that if

$$\max_{i,j=1,\ldots,N} \left\| \int_0^t \frac{\partial v_i}{\partial \xi_j}(\cdot, s)ds \right\|_{L_\infty(\Omega)} < \sigma \quad (0 < t < T), \tag{1.6}$$

then A is invertible, that is, $\det A \neq 0$. Thus, we have $\nabla_x = A^{-1}\nabla_\xi = (\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla\mathbf{v}(\xi, s)ds))\nabla_\xi$, where $\mathbf{V}_0(\mathbf{K})$ is an $N \times N$ matrix of C^∞ functions with respect to $\mathbf{K} = (k_{ij})$ for $|\mathbf{K}| < 2\sigma$ and $\mathbf{V}_0(0) = 0$. Here and hereafter, k_{ij} denote corresponding variables to $\int_0^t (\frac{\partial v_i}{\partial \xi_j})(\cdot, s)ds$. Let $\mathbf{n}$ be the unit outward normal to

Γ_0, and then we have

$$\mathbf{n}_t = \frac{A^{-1}\mathbf{n}}{|A^{-1}\mathbf{n}|}. \tag{1.7}$$

Suppose that $\rho(x,t)$, $\tau(x,t)$ and $\mathbf{u}(x,t)$ are solutions of (1.1). Setting $\rho(\mathbf{X}_\mathbf{v}(\xi,t),t) = \rho_* + \theta_0(\xi) + \theta(\xi,t)$ and $\tau = \tau_0(\xi) + \pi(\xi,t)$, we see that problem (1.1) is transformed to the following equations:

$$\begin{cases} \theta_t + (\rho_* + \theta_0)\mathrm{div}\,\mathbf{v} = F(\theta,\mathbf{v},\pi) & \text{in } \Omega\times(0,T), \\ (\rho_*+\theta_0)\mathbf{v}_t - \mathrm{Div}\,S(\mathbf{v}) + \nabla(P'(\rho_*+\theta_0)\theta) = \mathbf{g} + \beta\,\mathrm{Div}\,\pi + \mathbf{G}(\theta,\mathbf{v},\pi) & \text{in } \Omega\times(0,T), \\ \pi_t + \gamma\pi - g_\alpha(\nabla\nu,\tau_0) - \delta\mathbf{D}(\mathbf{v}) = -\gamma\tau_0 + \mathbf{L}(\theta,\mathbf{v},\pi) & \text{in } \Omega\times(0,T), \\ (\mathbf{S}(\mathbf{v}) - P'(\rho_*+\theta_0)\theta\mathbf{I} + \beta\pi)\mathbf{n} = \mathbf{h} + \mathbf{H}(\theta,\mathbf{v},\pi) & \text{on } \Gamma_1\times(0,T), \\ \mathbf{v} = 0 & \text{on } \Gamma_0\times(0,T), \\ (\theta,\mathbf{v},\pi)|_{t=0} = (0,\mathbf{u}_0,0) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $\mathbf{g} = -P'(\rho_*+\theta_0)\nabla\theta_0 + \beta\,\mathrm{Div}\,\tau_0$ and $\mathbf{h} = \big(P(\rho_*+\theta_0) - P(\rho_*)\big)\mathbf{n} - \beta\tau_0\mathbf{n}$. Moreover, $F(\theta,\mathbf{v})$, $\mathbf{G}(\mathbf{v},\theta,\pi)$, $\mathbf{L}(\mathbf{v},\pi)$, and $\mathbf{H}(\mathbf{v},\theta,\pi)$ are nonlinear functions of the forms:

$$F(\theta,\mathbf{v}) = -\theta\mathrm{div}\,\mathbf{v} - (\rho_*+\theta_0+\theta)V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}, ds\right)\nabla\mathbf{v},$$

$$\begin{aligned}
\mathbf{G}(\mathbf{v},\theta,\pi) &= -\theta\mathbf{v}_t + \mathrm{Div}\left(\mu V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} + (\nu-\mu)V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\mathbf{I}\right) \\
&\quad + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\left(\mu\left(D(\mathbf{v}) + V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right) + (\nu-\mu)\left(\mathrm{div}\,\mathbf{v} + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right)\mathbf{I}\right) \\
&\quad - P'(\rho_*+\theta_0+\theta)V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla(\theta_0+\theta) + \beta V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\tau_0 + \beta V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\pi \\
&\quad - \nabla\left(\int_0^1 P''(\rho_*+\theta_0+\ell\theta)(1-\ell)\,d\ell\theta^2\right),
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}(\mathbf{v},\theta,\pi) &= -\left\{\mu V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} + (\nu-\mu)\left(V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right)\mathbf{I}\right\}\mathbf{n} \\
&\quad - \left\{\mu\left(D(\mathbf{v}) + V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right) + (\nu-\mu)\left(\mathrm{div}\,\mathbf{v} + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right)\mathbf{I}\right\} \\
&\quad \times V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n} + \left(\int_0^1 P''(\rho_*+\theta_0+\ell\theta)(1-\ell)d\ell\theta^2\right)\mathbf{n} + (P(\rho_*+\theta_0+\theta) - P(\rho_*)) \\
&\quad \times V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n} - \beta(\tau_0+\pi)V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}(\mathbf{v},\pi) &= \mathbf{W}(\mathbf{v})\tau + V_W\left(\int_0^t \nabla\mathbf{v}ds\right)\nabla\mathbf{v}(\tau_0+\tau) - \tau\mathbf{W}(\mathbf{v}) - (\tau+\tau_0)V_W\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} \\
&\quad + \alpha(\tau\mathbf{D}(\mathbf{v}) + (\tau+\tau_0)V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} + \mathbf{D}(\mathbf{v})\tau + V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}(\tau+\tau_0)),
\end{aligned}$$

and $V_D(\mathbf{K})$, $V_W(\mathbf{K})$, and $V_{\mathrm{div}}(\mathbf{K})$ are some matrices of C^∞ functions with respect to $\mathbf{K}$ for $|\mathbf{K}| \le 2\sigma$, which satisfy the condition:

$$V_D(0) = 0, \qquad V_W(0) = 0, \qquad V_{\mathrm{div}}(0) = 0. \tag{1.9}$$

Employing the argumentation due to Ströhmer [32], we can show eventually that the correspondence $x = X_v(\xi,t)$ is invertible, then problem (1.1) and problem (1.8) are equivalent. Thus, we show the local well-posedness of problem (1.8).

To this end, the main step is to prove the L_p–L_q maximal regularity for the following linearized problem:

$$\begin{cases} \partial_t\rho + \gamma_1 \mathrm{div}\,\mathbf{u} = f & \text{in } \Omega\times(0,T),\\ \gamma_2\partial_t\mathbf{u} - \mathrm{Div}\,\mathbf{T}(\mathbf{u},\gamma_3\rho) = \delta_1\mathrm{Div}\,\tau + \mathbf{g} & \text{in } \Omega\times(0,T),\\ \partial_t\tau + \delta_2\tau - g_\alpha(\nabla\mathbf{u},\tau_1) = \delta_3\mathbf{D}(\mathbf{u}) + \mathbf{h} & \text{in } \Omega\times(0,T),\\ (\mathbf{T}(\mathbf{u},\gamma_3\rho) + \delta_1\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma_1,\\ \mathbf{u} = 0 & \text{on } \Gamma_0,\\ (\rho,\mathbf{u},\tau)|_{t=0} = (\rho_0,\mathbf{u}_0,\tau_0) & \text{in } \Omega, \end{cases} \tag{1.10}$$

where γ_1, γ_2, γ_3 and τ_1 are uniformly continuous functions with respect to $x \in \overline{\Omega}$, which satisfy the assumptions:

$$\begin{aligned} &\rho_*/2 \le \gamma_2(x) \le 2\rho_*, \quad 0 \le \gamma_1(x), \qquad \gamma_3(x) \le \rho_1, \qquad \|\nabla\gamma_\ell\|_{L_r(\Omega)} \le \rho_1, \quad (\ell = 1,2,3),\\ &\|\tau_1\|_{W^1_r(\Omega)} \le \rho_1, \end{aligned} \tag{1.11}$$

while δ_1, δ_2, and δ_3 are positive constants. Note that in problem (1.1) we have written $\delta_1 = \beta$, $\delta_2 = \gamma$ and $\delta_3 = \delta$.

The maximal L_p regularity was proved by Solonnikov [26,27] for the general parabolic equations which satisfy the uniform Lopatinski–Shapiro conditions. After Solonnikov's study about the maximal regularity, to obtain the maximal L_p regularity result in the model problem, Moglievskii [10,11], Mucha and Zajaczkowski [12] and Solonnikov [29] used the Marcinkiewicz–Mikhlin–Lizorkin multiplier theorems together with some Hardy type inequality. Prüss and Simonett [15,16] used $\mathcal{H}^\infty$ calculus and Shibata–Shimizu [25] used the $\mathcal{R}$-boundedness and the Weis operator valued Fourier multiplier theorem.

On the other hand, Denk, Hieber and Prüss [5], Shibata [21], Enomoto and Shibata [6], Enomoto, von Below and Shibata [7], Dario and Shibata [8], Murata [13] used another methods, namely they construct the $\mathcal{R}$ bounded solution operator to the resolvent problem and used the Weis operator valued Fourier multiplier theorem to obtain the maximal L_p in time and L_q in space regularity. In this paper, we follow Enomoto, von Below, and Shibata [6,7] to prove the maximal regularity result for problem (1.10) with help of the $\mathcal{R}$ bounded operator for the generalized resolvent problem:

$$\begin{cases} \lambda\rho + \gamma_1 \mathrm{div}\,\mathbf{u} = f & \text{in } \Omega,\\ \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{T}(\mathbf{u},\gamma_3\rho) = \delta_1\mathrm{Div}\,\tau + \mathbf{g} & \text{in } \Omega,\\ \lambda\tau + \delta_2\tau - g_\alpha(\nabla\mathbf{u},\tau_1) = \delta_3\mathbf{D}(\mathbf{u}) + \mathbf{h} & \text{in } \Omega,\\ (\mathbf{T}(\mathbf{u},\gamma_3\rho) + \delta_1\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma_1,\\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases} \tag{1.12}$$

1.1. Notation and the definition of uniform domains

Before stating our main result, we introduce the notation used throughout the paper, and some definitions. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y, and $\mathrm{Hol}\,(U,\mathcal{L}(X,Y))$ the set of all $\mathcal{L}(X,Y)$ valued holomorphic functions defined on a complex domain U. For any domain D and $1 \le p,q \le \infty$, $L_q(D)$, $W^m_q(D)$ and $B^s_{p,q}(D)$ denote the usual Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W^m_q(D)}$ and $\|\cdot\|_{B^s_{q,p}(D)}$ denote their norms, respectively. We set $W^0_q(D) = L_q(D)$, $W^s_q(D) = B^s_{q,q}(D)$ and

$$W^{m,\ell}_q(D) = \{(f,\mathbf{g},\mathbf{h}) \mid f \in W^m_q(D),\ \mathbf{g} \in W^\ell_q(D)^N,\ \mathbf{h} \in W^m_q(D)^{N^2}\}.$$

$C^\infty_0(D)$ denotes the set all $C^\infty(\mathbb{R}^N)$ functions whose supports are compact and contained in D. We set $(f,g)_D = \int_D f(x)g(x)dx$. $L_p((a,b),X)$ and $W^m_p((a,b),X)$ denote the usual Lebesgue space and Sobolev space of X-valued function defined on an interval (a,b), while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W_p^m((a,b),X)}$ denote their norms, respectively. The d-product space of X is defined by $X^d = \{f = (f_1,\ldots,f_d) \mid f_i \in X\ (i = 1,\ldots,d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural numbers, real numbers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For any multi-index $\kappa = (\kappa_1,\ldots,\kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1+\cdots+\kappa_N$ and $\partial_x^k = \partial_1^{\kappa_1}\cdots\partial_N^{\kappa_N}$ with $x = (x_1,\ldots,x_N)$ and $\partial_j = \partial/\partial x_j$. For scalar function f and N-vector of functions $\mathbf{g}$, we set

$$\nabla f = (\partial_1 f,\ldots,\partial_N f), \qquad \nabla\mathbf{g} = (\partial_i g_j \mid i,j = 1,\ldots,N),$$
$$\nabla^2 f = (\partial^\alpha f \mid |\alpha| = 2), \qquad \nabla^2\mathbf{g} = (\partial^\alpha g_i \mid |\alpha| = 2,\ i = 1,\ldots,N).$$

For $\mathbf{a} = (a_1\ldots,a_N)$ and $\mathbf{b} = (b_1\ldots,b_N)$, we set $\mathbf{a}\cdot\mathbf{b} = \langle\mathbf{a},\mathbf{b}\rangle = \sum_{j=1}^N a_j b_j$. For scalar functions f,g and N-vectors of functions $\mathbf{f}$, $\mathbf{g}$ we set $(f,g) = \int_D f(x)g(x)dx$ and $(\mathbf{f},\mathbf{g})_D = \int_D \mathbf{f}(x)\cdot\mathbf{g}(x)dx$. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a,b,\ldots$. The values of constants C and $C_{a,b,\ldots}$ may change from line to line. We use the bold-face letters to denote N-vector valued function and $N\times N$ matrix of functions. And also, we use the Greek letters to denote mass density as well as elastic tensor.

Next, we introduce a definition.

Definition 1.1. Let $1 < r < \infty$ and let Ω be a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We say that Ω is a uniform $W_r^{2-1/r}$ domain, if there exists positive constants α,β and K such that for any $x_0 = (x_{01},\ldots,x_{0N}) \in \partial\Omega$ there exist a coordinate number j and a $W_r^{2-1/r}$ function $h(x')$ $(x' = (x_1,\ldots,\hat{x}_j,\ldots,x_N))$ defined on $B'_\alpha(x'_0)$ with $x'_0 = (x_{01},\ldots,\hat{x}_{0j},\ldots,x_{0N})$ and $\|h\|_{W_r^{2-1/r}(B'_\alpha(x'_0))} \le K$ such that

$$\begin{aligned}\Omega\cap B_\beta(x_0) &= \{x\in\mathbb{R}^N \mid x_j > h(x')\ (x'\in B'_\alpha(x'_0))\}\cap B_\beta(x_0)\\ \partial\Omega\cap B_\beta(x_0) &= \{x\in\mathbb{R}^N \mid x_j = h(x')\ (x'\in B'_\alpha(x'_0))\}\cap B_\beta(x_0).\end{aligned} \tag{1.13}$$

Here, $(x_1,\ldots,\hat{x}_j,\ldots,x_N) = (x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N)$, $B'_\alpha(x'_0) = \{x'\in\mathbb{R}^{N-1} \mid |x'-x'_0| < \alpha\}$ and $B_\beta(x_0) = \{x\in\mathbb{R} \mid |x-x_0| < \beta\}$.

1.2. Main results

The following theorem represents the main result of this paper.

Theorem 1.2. *Let $N < q < \infty$, $2 < p < \infty$ and $R > 0$. Then, there exists a time $T > 0$ depending on R such that if the initial data $(\theta_0,\mathbf{u}_0,\tau_0)$ for Eqs. (1.1) satisfy*

$$\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-\frac{1}{p})}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \le R, \tag{1.14}$$

the range condition:

$$\frac{\rho_*}{2} < \rho_* + \theta_0 < 2\rho_*, \tag{1.15}$$

and the compatibility condition:

$$(\mathbf{T}(\mathbf{u}_0, P(\rho_*+\theta_0)) + \beta\tau_0)\mathbf{n} = -P(\rho_*)\mathbf{n} \quad \text{on } \Gamma_1, \qquad \mathbf{u}_0 = 0 \quad \text{on } \Gamma_0, \tag{1.16}$$

then problem (1.8) admits a unique solution $(\theta,\mathbf{v},\pi)$ with

$$\theta\in W_p^1((0,T),W_q^1(\Omega)), \qquad \mathbf{v}\in W_p^1((0,T),L_q(\Omega))\cap L_p((0,T),W_q^2(\Omega)), \qquad \pi\in W_p^1((0,T),W_q^1(\Omega))$$

satisfying the conditions:

$$\frac{\rho_*}{4} < \rho_* + \theta_0 < 4\rho_*, \qquad \max_{i,j=1,\ldots,n}\int_0^T \|(\partial u_i/\partial\xi_j)(\cdot,s)ds\|_{L_\infty(\Omega)} < \sigma,$$

and the estimate:

$$\|\theta\|_{W^1_p((0,T),W^1_q(\Omega))} + \|\mathbf{v}\|_{W^1_p((0,T),L_q(\Omega))} + \|\mathbf{v}\|_{L_p((0,T),W^2_q(\Omega))} + \|\pi\|_{W^1_p((0,T),W^1_q(\Omega))} \le CR$$

with some constant C independent of R.

Using the argumentation due to Ströhmer [32], we see that the map $x = \mathbf{X}_{\mathbf{v}}(\xi, t)$ is a diffeomorphism with suitable regularity, so that for problem (1.1) by Theorem 1.2 we have

Theorem 1.3. *Let $N < q < \infty$, $2 < p < \infty$ and $R > 0$. Then, there exists a time $T_1 > 0$ depending on R such that if the initial data $(\theta_0, \mathbf{u}_0, \tau_0)$ for problem (1.1) satisfies the same condition as in Theorem 1.2, then problem (1.1) admits a unique solution $(\rho, \mathbf{u}, \tau)$ with*

$$\rho - \rho_* \in W^1_p((0,T), L_q(\Omega_t)) \cap L_p((0,T), W^1_q(\Omega_t)), \quad \mathbf{u} \in W^1_p((0,T), L_q(\Omega_t)) \cap L_p((0,T), W^2_q(\Omega_t)),$$
$$\tau \in W^1_p((0,T), L^1_q(\Omega_t)) \cap L_p((0,T), W^1_q(\Omega_t)).$$

Remark 1.4. In Theorem 1.3, $v \in W^\ell_p((0,T), W^m_q(\Omega_t))$ means that $\partial^j_t v \in W^m_q(\Omega_t)$ for $t \in (0,T)$ and $j = 0, 1, \ldots, \ell$, where $W^0_p = L_p$, $W^0_q = L_q$ and $\partial^0 v = v$, and

$$\|v\|_{W^\ell_p((0,T),W^m_q(\Omega_t))} = \sum_{j=0}^{\ell}\Big(\int_0^T (\|\partial^j_t v(\cdot,t)\|_{W^m_q(\Omega_t)})^p \, ds\Big)^{1/p} < \infty.$$

Including this introduction, we organize the paper as follows. In Section 2, we discuss the extension of the unit outer normal $\mathbf{n}$ to the whole space, give some proposition about uniform $W^{2-1/r}_r$ domains and prepare some calculus lemmas for the latter use. In Section 3, we show the existence of $\mathcal{R}$-bounded solution operator to problem (1.12) and (1.10). In Section 4, we state the existence of $\mathcal{R}$-bounded solution operator for problem (1.12) and we prove the maximal regularity result for problem. In Section 5, we prove Theorem 1.2.

2. Some properties of the uniform $W^{2-1/r}_r$ domain

In this section, we discuss some properties of the uniform $W^{2-1/r}_r$ domain and we prepare some calculus lemmas for the latter use. Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. We assume that $\nabla\Phi$ and $\nabla\Phi^{-1}$ have the forms: $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(x)$, where $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_{-1}(x)$ are matrices of functions in $W^2_r(\mathbb{R}^N)$ with $N < r < \infty$ such that

$$\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_1, \qquad \|\nabla(B, B_{-1})\|_{L_r(\mathbb{R}^N)} \le M_2. \tag{2.1}$$

Let $\mathcal{A}_{ij}$, $\mathcal{A}_{-ij}$, $B_{ij}(x)$ and $B_{-1ij}(\xi)$ be the (i,j) elements of $\mathcal{A}$, $\mathcal{A}_-$, $B(x)$ and $B_{-1}(x)$, respectively. We will choose M_1 small enough eventually, so that in the sequel, we may assume that $0 < M_1 \le 1 \le M_2$. Let $\Omega_+ = \Phi(\mathbb{R}^N_+)$ and $\Gamma_+ = \Phi(\mathbb{R}^N_0)$, where

$$\mathbb{R}^N_+ = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}, \qquad \mathbb{R}^N_0 = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}.$$

The Γ_+ is the boundary of Ω_+ and represented by $\xi = \Phi(x', 0)$ with $x' = (x_1, \ldots, x_{N-1})$. Let

$$N_i = \det\begin{pmatrix} \partial_1\xi_1 & \cdots & \partial_{N-1}\xi_1 \\ \vdots & \cdots & \vdots \\ \partial_1\xi_{i-1} & \cdots & \partial_{N-1}\xi_{i-1} \\ \partial_1\xi_{i+1} & \cdots & \partial_{N-1}\xi_{i+1} \\ \vdots & \cdots & \vdots \\ \partial_1\xi_N & \cdots & \partial_{N-1}\xi_N \end{pmatrix} \quad \text{with } \partial_i\xi_j = \frac{\partial\Phi_j(x)}{\partial x_i}, \tag{2.2}$$

where $\Phi = (\Phi_1, \ldots, \Phi_N)$, let $\tilde{n}_{+i} = (-1)^{N+i} N_i / \sqrt{\sum_{k=1}^N N_k^2}$, let $n_{+i} = \tilde{n}_{+i} \circ \Phi^{-1}$, and let $\mathbf{n}_{\Gamma_+} = (n_{+1}, \ldots, n_{+N})$. We see that $\mathbf{n}_{\Gamma_+}|_{\Gamma_+}$ is the unit outer normal to Γ_+. Moreover, $\mathbf{n}_{\Gamma_+}$ is defined on $\mathbb{R}^N$ and by (2.1)

$$\|\mathbf{n}_{\Gamma_+}\|_{L_\infty(\mathbb{R}^N)} \leq C_N, \qquad \|\nabla \mathbf{n}_{\Gamma_+}\|_{W_q^1(\mathbb{R}^N)} \leq C_{M_2}. \tag{2.3}$$

Several properties of uniform $W_r^{2-1/r}$ domains are given in the following proposition which was proved in Enomoto and Shibata [6, Proposition 6.1].

Proposition 2.1. *Let $N < r < \infty$ and let Ω be a uniform $W_r^{2-1/r}$ domain in $\mathbb{R}^N$. Let M_1 be any small number $\in (0,1)$. Then, there exist constants $M_2 > 0$, $0 < d^0, d^1, d^2 < 1$, an open set U, at most countably many N-vector of functions Φ_j^0 and Φ_j^1, and points $x_j^0 \in \Gamma_0$, $x_j^1 \in \Gamma_1$ and $x_j^2 \in \Omega$ such that the following assertions hold:*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbb{R}^N$ $(i = 0,1)$ are bijective of C^1 class.*
(ii) $\Omega = \left(\bigcup_{i=0}^1 \bigcup_{j=1}^\infty (\Phi_j^i(\mathbb{R}_+^N) \cap B_{d^i}(x_j^i))\right) \cup \left(\bigcup_{j=1}^\infty B_{d^2}(x_j^2)\right)$, $B_{d^2}(x_j^2) \subset \Omega$, $\Phi_j(\mathbb{R}_+^N) \cap B_{d^i}(x_j^i) = \Omega \cap B_{d^i}(x_j^i)$ $(i = 0,1)$, $\Phi_j^i(\mathbb{R}_0^N) \cap B_{d^i}(x_j^i) = \Gamma_i \cap B_{d^i}(x_j^i)$ $(i = 0,1)$.
(iii) *There exist C^∞ functions ζ_j^i and $\tilde{\zeta}_j^i (i = 0,1,2,\ j = 1,2,3,\ldots)$ such that*

$$0 \leq \zeta_j^i, \qquad \tilde{\zeta}_j^i \leq 1, \quad \operatorname{supp} \zeta_j^i,\ \operatorname{supp} \tilde{\zeta}_j^i \subset B_{d^i}(x_j^i), \qquad \|\zeta_j^i\|_{W_\infty^2(\mathbb{R}^N)},\ \|\tilde{\zeta}_j^i\|_{W_\infty^2(\mathbb{R}^N)} \leq c_0,$$
$$\tilde{\zeta}_j^i = 1 \quad \text{on } \operatorname{supp} \zeta_j^i, \qquad \sum_{i=0}^2 \sum_{j=1}^\infty \zeta_j^i = 1 \quad \text{on } \overline{\Omega}, \qquad \sum_{j=1}^\infty \zeta_j^i = 1 \quad \text{on } \Gamma_i\ (i = 0,1).$$

Here, c_0 is a constant which depends on M_2, N, q and r, but is independent of $j = 1,2,3,\ldots$.
(iv) $\nabla \Phi_j^i = \mathcal{A}_j^i + B_j^i$, $\nabla(\Phi_j^i)^{-1} = \mathcal{A}_{j,-}^i + B_{j,-}^i$, *where $\mathcal{A}_j^i$ and $\mathcal{A}_{j,-}^i$ are $N \times N$ constant orthonormal matrices, and B_j^i and $B_{j,-}^i$ are $N \times N$ matrices of $W_r^{1+i}(\mathbb{R}^N)$ functions defined on $\mathbb{R}^N$ which satisfy the conditions: $\|B_j^i\|_{L_\infty(\mathbb{R}^N)} \leq M_1$, $\|B_{j,-}^i\|_{L_\infty(\mathbb{R}^N)} \leq M_1$, $\|\nabla B_j^i\|_{L_r(\mathbb{R}^N)} \leq M_2$ and $\|\nabla B_{j,-}^i\|_{L_r(\mathbb{R}^N)} \leq M_2$ for $i = 0,\ 1$ and $j = 1,2,3,\ldots$.*
(v) *There exists a natural number $L \geq 2$ such that any $L+1$ distinct sets of $\{B_{d^i(x_j^i)} \mid i = 0,1,2,\ j = 1,2,3,\ldots\}$ have an empty intersection.*

By Proposition 2.1(v), we have

$$C_q^1 \|f\|_{L_q(\Omega)}^q \leq \sum_{i=0}^2 \sum_{j=1}^\infty \|\zeta_j^i f\|_{L_q(\Omega)}^q \leq \sum_{i=0}^2 \sum_{j=1}^\infty \|f\|_{L_q(\Omega \cap B_j^1)}^q \leq C_q^2 \|f\|_{L_q(\Omega)}^q \tag{2.4}$$

for any $f \in L_q(\Omega)$ and $1 \leq q < \infty$ with some positive constants C_q^1 and C_q^2.

In the sequel, we write $B_j^i = B_{d^i}(x_j^i)$, $(\Phi_j^i)^{-1} = \Psi_j^i$, $\Omega_j^1 = \Phi_j^1(\mathbb{R}_+^N)$, and $\Gamma_j^1 = \Phi_j^1(\mathbb{R}_0^N)$ for the sake of simplicity. The Γ_j^1 is the boundary of Ω_j^1. We introduce some properties of the unit outer normal $\mathbf{n}$ to Γ_1, the extension operator $\mathbf{E}$, the space $\mathbf{W}_q^{-1}(\Omega)$ and its norm $\|\cdot\|_{\mathbf{W}_q^{-1}(\Omega)}$, and we prove some inequalities for the later use. From the consideration at the beginning of this section it follows the existence of $\mathbf{n}_k^1 \in W_{r,\text{loc}}^1(\mathbb{R}^N)$ such that $\mathbf{n}_k^1 = \mathbf{n}$ on $\Gamma_1 \cap B_k^1$ and

$$\|\mathbf{n}_k^1\|_{W_r^1(B_k^1)} \leq C. \tag{2.5}$$

Let $\tilde{\mathbf{n}} = \sum_{k=1}^\infty \zeta_k^1 \mathbf{n}_k^1$ and $\mathcal{S} = \cup_{k=1}^\infty \operatorname{supp} \zeta_k^1$, and then $\mathbf{n} = \tilde{\mathbf{n}}$ on Γ_1 and $\operatorname{supp} \tilde{\mathbf{n}} \subset \mathcal{S}$. For the notational simplicity, hereinafter we write $\tilde{\mathbf{n}} = \sum_{k=1}^\infty \zeta_k^1 \mathbf{n}_k^1$. Since $\tilde{\mathbf{n}} = \mathbf{n}$ on Γ_1, we write $\mathbf{n} = \tilde{\mathbf{n}}$ unless confusion may occur.

Next, let p_j $(j = 1, 2, 3, 4)$ be numbers such that $\sum_{j=1}^4 (-j)^k p_j = 1$ for $k = -1, 0, 1, 2$. Given function $f \in L_{1,\mathrm{loc}}(\mathbb{R}^N_+)$, let

$$\iota[f](x) = \begin{cases} f(x', x_N) & (x_N > 0), \\ \displaystyle\sum_{j=1}^4 p_j f(x', -jx_N) & (x_N < 0). \end{cases}$$

Obviously, $\partial_N^k \iota[f]_{x_N=0+} = \partial_N^k \iota[f]_{x_N=0-} = (\partial_N f)(x', 0+)$, so that $\|\iota[f]\|_{W_q^k(\mathbb{R}^N)} \le C\|f\|_{W_q^k(\mathbb{R}^N_+)}$ for $k = 0, 1, 2$, where $W_q^0 = L_q$. Moreover, $\iota[\partial_N f] = \partial_N(\sum_{k=1}^4 (-j)^{-1} p_j f(x', -jx_N))$ for $x_N < 0$ and $\sum_{k=1}^4 (-j)^{-1} p_j f(x', -jx_N)|_{x_N=0-} = f(x', 0+)$, so that $\|\iota[\partial_N f]\|_{W_q^{-1}(\mathbb{R}^N)} \le C\|f\|_{L_q(\Omega)}$, where $W_q^{-1}(\mathbb{R}^N)$ is the dual space of $W_q^1(\mathbb{R}^N)$.

Let the extension operator $\mathbf{E}$ be defined by

$$\mathbf{E}[f] = \sum_{i=0}^1 \sum_{j=1}^\infty \iota[(\zeta_j^i f) \circ \Phi_j^i] \circ \Psi_j^i + \sum_{j=1}^\infty \zeta_j^2 f.$$

For the product fg, $\mathbf{E}[fg]$ is defined by $\mathbf{E}[fg] = \mathbf{E}[f]\mathbf{E}[g]$, and if g is defined on $\mathbb{R}^N$, $\mathbf{E}[fg]$ is defined by $\mathbf{E}[fg] = \mathbf{E}[f]g$. Obviously, $\mathbf{E}[f] = f$ in Ω. Moreover, we have

$$\begin{aligned} \|\mathbf{E}[u]\|_{W_q^k(\mathbb{R}^N)} &\le C\|u\|_{W_q^k(\Omega)} \quad \text{for } k = 0, 1, 2, \\ \|\mathbf{E}[\nabla u]\|_{W_q^{-1}(\mathbb{R}^N)} &\le C\|u\|_{L_q(\Omega)}. \end{aligned} \tag{2.6}$$

Let

$$\mathbf{W}_q^{-1}(\Omega) = \{f \in L_{1,\mathrm{loc}}(\Omega) \mid \mathbf{E}[f] \in W_q^{-1}(\mathbb{R}^N)\}, \qquad \|f\|_{\mathbf{W}_q^{-1}(\Omega)} = \|\mathbf{E}[f]\|_{W_q^{-1}(\mathbb{R}^N)}.$$

For the later use, we prove

Lemma 2.2. *Let $1 < q < \infty$ and $N < s < \infty$. Assume that $\max(q, q') \le s$. Then, the following assertions hold.*

(1)

$$\|fg\|_{W_q^1(\Omega)} \le C\|f\|_{W_q^1(\Omega)}\|g\|_{W_s^1(\Omega)}, \qquad \|g\|_{L_\infty(\Omega)} \le C\|g\|_{W_s^1(\Omega)}.$$

(2)

$$\begin{aligned} \|\nabla u\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C\|u\|_{L_q(\Omega)}, \\ \|uv\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C_q\|u\|_{\mathbf{W}_q^{-1}(\Omega)}\|v\|_{W_s^1(\Omega)}, \\ \|uv\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C_q\|u\|_{L_q(\Omega)}\|v\|_{L_s(\Omega)}. \end{aligned}$$

(3) *Let g_k $(k = 1, 2, \ldots)$ be functions in $W^1_{s,\mathrm{loc}}(\mathbb{R}^N)$ such that*

$$\operatorname{supp} g_k \subset B_k^1, \qquad \|g_k\|_{W_s^1(B_k^1)} \le \gamma_0, \tag{2.7}$$

for some constant γ_0 independent of $k = 1, 2, 3, \ldots$. Then,

$$\begin{aligned} \Big\|\sum_{k=1}^\infty \zeta_k^1 f g_k\Big\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C_q\gamma_0\|f\|_{\mathbf{W}_q^{-1}(\Omega)}, \\ \Big\|\sum_{k=1}^\infty \zeta_k^1 f g_k\Big\|_{W_q^k(\Omega)} &\le C_q\gamma_0\|f\|_{W_q^k(\Omega)} \quad (k = 0, 1). \end{aligned}$$

Proof. (1) It follows from the Sobolev imbedding theorem that $\|g\|_{L_\infty(\Omega)} \leq C\|g\|_{W^1_s(\Omega)}$, so that we also have $\|(f, \nabla f)g\|_{L_q(\omega)} \leq C\|f\|_{W^1_q(\Omega)}\|g\|_{W^1_s(\Omega)}$. By the Sobolev imbedding theorem, we have

$$\|fg\|_{L_a(\Omega)} \leq C\|f\|_{L_s(\Omega)}\|g\|_{W^1_a(\Omega)} \quad (a = q, q'). \tag{2.8}$$

In fact, by the Hölder inequality, we have $\|fg\|_{L_a(\Omega)} \leq C\|f\|_{L_s(\Omega)}\|g\|_{L_b(\Omega)}$ with $1/a = 1/s + 1/b$. Note that $a \leq s$. If $a = s$, then $b = \infty$ and $N < a < \infty$, so that by the Sobolev imbedding theorem $\|g\|_{L_b(\Omega)} \leq C\|g\|_{W^1_a(\Omega)}$. If $a < s$, then $N(1/a - 1/b) = N/s < 1$, so that by the Sobolev imbedding theorem we also have $\|g\|_{L_b(\Omega)} \leq C\|g\|_{W^1_a(\Omega)}$. Thus, we have (2.8).

Applying (2.8), we have $\|f\nabla g\|_{L_q(\Omega)} \leq C\|f\|_{W^1_q(\Omega)}\|\nabla g\|_{L_s(\Omega)}$. Summing up, we have shown the assertion (1).

(2) The first inequality follows from (2.6). To prove the second one, we observe that

$$|(\mathbf{E}[uv], \varphi)_{\mathbb{R}^N}| \leq \|u\|_{\mathbf{W}^{-1}_q(\Omega)}\|\mathbf{E}[v]\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$. By (2.8) we have

$$\|(\nabla\mathbf{E}[v])\varphi\|_{L_{q'}(\mathbb{R}^N)} \leq C\|\nabla\mathbf{E}[v]\|_{L_s(\mathbb{R}^N)}\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)}. \tag{2.9}$$

Thus, we have $\|\mathbf{E}[v]\varphi\|_{W^1_{q'}(\mathbb{R}^N)} \leq C\|\mathbf{E}[v]\|_{W^1_s(\mathbb{R}^N)}\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$, which implies the second inequality. Analogously, using Hölder's inequality and replacing $\nabla\mathbf{E}[v]$ by $\mathbf{E}[v]$ in (2.9), we have

$$|(\mathbf{E}[uv], \varphi)_{\mathbb{R}^N}| \leq \|\mathbf{E}[u]\|_{L_q(\mathbb{R}^N)}\|\mathbf{E}[v]\varphi\|_{L_{q'}(\mathbb{R}^N)} \leq C\|\mathbf{E}[u]\|_{L_q(\mathbb{R}^N)}\|\mathbf{E}[v]\|_{L_s(\mathbb{R}^N)}\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)},$$

which implies the last inequality.

(3) To prove the first inequality, setting $g = \sum_{k=1}^\infty \zeta^1_k g_k$, we observe that

$$|(\mathbf{E}[fg], \varphi)_{\mathbb{R}^N}| = |(\mathbf{E}[f], g\varphi)_{\mathbb{R}^N}| \leq \|f\|_{\mathbf{W}^{-1}_q(\Omega)}\|g\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$. By (2.4) replacing Ω by $\mathbb{R}^N$, (2.7) and (2.9), we have

$$\begin{aligned}
\|\nabla(g\varphi)\|^{q'}_{L_{q'}(\mathbb{R}^N)} &\leq C_{N,q'}\sum_{k=1}^\infty \|\nabla(\zeta^1_k g_k\tilde{\zeta}^1_k\varphi)\|^{q'}_{L_{q'}(\mathbb{R}^N)} \\
&\leq C_{N,q'}\sum_{k=1}^\infty(\|\nabla(\zeta^1_k g_k)\|^{q'}_{L_s(\mathbb{R}^N)}\|\tilde{\zeta}^1_k\varphi\|^{q'}_{W^1_{q'}(\mathbb{R}^N)} + \|\zeta^1_k g_k\|^{q'}_{L_\infty(\mathbb{R}^N)}\|\tilde{\zeta}^1_k\varphi\|^{q'}_{W^1_{q'}(\mathbb{R}^N)}) \\
&\leq C_{N,q'}\gamma_0^{q'}\sum_{k=1}^\infty \|\varphi\|^{q'}_{W^1_{q'}(B^1_k)} \leq C_{N,q'}\gamma_0^{q'}\|\varphi\|^{q'}_{W^1_{q'}(\mathbb{R}^N)}.
\end{aligned}$$

Analogously, we also have $\|g\varphi\|_{L_{q'}(\mathbb{R}^N)} \leq C_{N,q'}\gamma_0\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$. Thus, we have the first inequality.

Analogously, by (2.4) we easily have the second inequalities, which complete the proof of Lemma 2.2. □

For example, using (2.5) and Lemma 2.2 we have

$$\begin{aligned}
&\|f\mathbf{n}\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}, \qquad \|f\mathbf{n}\|_{W^1_q(\Omega)} \leq C\|f\|_{W^1_q(\Omega)}, \\
&\|fg\mathbf{n}\|_{\mathbf{W}^{-1}_q(\Omega)} \leq C\|f\|_{W^{-1}_q(\Omega)}\|g\|_{W^1_q(\Omega)}, \qquad \|fg\mathbf{n}\|_{\mathbf{W}^{-1}_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}\|g\|_{L_q(\Omega)}
\end{aligned} \tag{2.10}$$

with some constant $C > 0$.

3. $\mathcal{R}$ bounded solution operators

In this section, we prove the existence of $\mathcal{R}$ bounded solution operator associated with generalized resolvent problem (1.12). First of all, we introduce the definition of the $\mathcal{R}$ bounded operator family.

Definition 3.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, we have the inequality:

$$\left\{\int_0^1 \left\|\sum_{j=1}^n r_j(u) T_j x_j\right\|_Y^p du\right\}^{1/p} \leq C \left\{\int_0^1 \left\|\sum_{j=1}^n r_j x_j\right\|_X^p du\right\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

The resolvent parameter λ in problem (1.12) varies in $\Sigma_{\epsilon,\lambda_0}$ with

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \pi - \epsilon,\ |\lambda| \geq \lambda_0\} \quad (\epsilon \in (0, \pi/2), \lambda_0 > 0).$$

The main result for the $\mathcal{R}$ bounded solution operator is the following theorem.

Theorem 3.2. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \geq \max(q, q')$. Let Ω be a uniform $W_r^{2-1/r}$ domain and $\lambda \in \Sigma_{\epsilon,\lambda_0}$. Set*

$$X_q(\Omega) = \{(f, \mathbf{g}, \mathbf{h}, \mathbf{k}) \mid (f, \mathbf{g}, \mathbf{h}) \in W_q^{1,0}(\Omega), \mathbf{k} \in W_q^1(\Omega)^N\},$$
$$\mathcal{X}_q(\Omega) = \{(F_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5) \mid F_1 \in W_q^1(\Omega), \mathbf{F}_2 \in L_q(\Omega)^N, \mathbf{F}_3 \in L_q(\Omega)^N, \mathbf{F}_4 \in L_q(\Omega)^{N^2}, \mathbf{F}_5 \in W_q^1(\Omega)^{N^2}\}.$$

Then, there exists a $\lambda_0 \geq 1$ and an operator family $R(\lambda)$ with

$$R(\lambda) \in \mathrm{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,2}(\Omega)))$$

such that for any $(f, \mathbf{g}, \mathbf{h}, \mathbf{k}) \in X_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $(\rho, \mathbf{u}, \tau) = R(\lambda)(f, \mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k}, \mathbf{h})$ is a unique solution to problem (1.12).

Moreover, there exists a constant C such that

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau\partial\tau)^\ell(\lambda R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau\partial\tau)^\ell(\gamma R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^2})}(\{(\tau\partial\tau)^\ell(\lambda^{1/2}\nabla P_v R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^3})}(\{(\tau\partial\tau)^\ell(\nabla^2 P_v R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\leq C \quad (\ell = 0, 1),
\end{aligned} \tag{3.1}$$

with $\lambda = \gamma + i\tau$. Here, P_v is the projection operator defined by $P_v(\rho, \mathbf{u}, \tau) = \mathbf{u}$.

Remark 3.3. The F_1, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ and $\mathbf{F}_5$ are variables corresponding to f, $\mathbf{g}$, $\lambda^{1/2}\mathbf{k}$, $\nabla\mathbf{k}$, and $\mathbf{h}$, respectively.

In the sequel, we prove Theorem 3.2. To prove Theorem 3.2, we reduce the problem to the Lamé equation:

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) = \mathbf{g} & \text{in } \Omega,\\ \mathbf{S}(\mathbf{u})\mathbf{n} = \mathbf{k} & \text{on } \Gamma_1,\\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases} \tag{3.2}$$

According to Enomoto, von Below and Shibata [7], we know

Theorem 3.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \geq \max(q, q')$. Let Ω be a uniform $W_r^{2-1/r}$ domain. Let*

$$Y_q(\Omega) = \{(\mathbf{g}, \mathbf{k}) \mid \mathbf{g} \in L_q(\Omega)^N, \mathbf{k} \in W_q^1(\Omega)^N\},$$
$$\mathcal{Y}_q(\Omega) = \{(\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \mid \mathbf{F}_2 \in L_q(\Omega)^N, \mathbf{F}_3 \in L_q(\Omega)^N, \mathbf{F}_4 \in L_q(\Omega)^{N^2}\}.$$

Then there exist a $\lambda_0 \geq 1$ and an operator family $\mathcal{A}(\lambda)$ with

$$\mathcal{A}(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\Omega), W_q^2(\Omega)^N))$$

such that for any $(\mathbf{g}, \mathbf{k}) \in Y_q(\Omega)$ and $\lambda \in \Lambda_{\epsilon,\lambda_0}$, $\mathbf{u} = \mathcal{A}(\lambda)(\mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k})$ is a unique solution of problem (3.2) and $\mathcal{A}(\lambda)$ satisfy the estimates

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\tau\partial\tau)^\ell(G_\lambda\mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1)$$

with $\lambda = \gamma + i\tau$, where we set $\tilde{N} = 2N + N^2 + N^3$ and $G_\lambda\mathbf{u} = (\lambda\mathbf{u}, \gamma\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u})$.

Setting $\theta = \lambda^{-1}(f - \gamma_1 \mathrm{div}\,\mathbf{u})$ and $\tau = (\lambda + \delta_2)^{-1}\big(\delta_3 D(\mathbf{u}) + g_\alpha(\nabla\mathbf{u}, \tau_1) + \mathbf{h}\big)$ for the case $\lambda \neq 0$ in (1.12), we have

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) = \mathbf{g} - \lambda^{-1}\nabla(\gamma_3 f) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,\mathbf{h} \\ \qquad + \lambda^{-1}\nabla(\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u}) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})) & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n} = \mathbf{k} + (\lambda^{-1}\gamma_3 f - \delta_1(\lambda+\delta_2)^{-1}\mathbf{h})\mathbf{n} \\ \qquad - (\lambda^{-1}\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u} + \delta_1(\lambda+\delta_2)^{-1}(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})))\mathbf{n} & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases}$$

Thus, $\mathbf{g} - \lambda^{-1}\nabla(\gamma_3 f) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,\mathbf{h}$ and $\mathbf{k} + (\lambda^{-1}\gamma_3 f - \delta_1(\lambda+\delta_2)^{-1}\mathbf{h})\mathbf{n}$ being renamed $\mathbf{g}$ and $\mathbf{k}$, respectively, for the sake of simplicity, we consider the following equations:

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) - B_1(\lambda)(\mathbf{u}) = \mathbf{g} & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n} - B_2(\lambda)(\mathbf{u}) = \mathbf{k} & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \end{cases} \tag{3.3}$$

where we have set

$$\begin{aligned} B_1(\lambda)(\mathbf{u}) &= \lambda^{-1}\nabla(\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u}) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})), \\ B_2(\lambda)(\mathbf{u}) &= -(\lambda^{-1}\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u} + \delta_1(\lambda+\delta_2)^{-1}(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})))\mathbf{n}. \end{aligned} \tag{3.4}$$

To prove Theorem 3.2, we use the following two lemmas about the $\mathcal{R}$-norms.

Lemma 3.5 ([5]). *Let X, Y and Z be Banach space and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families*

1. *If X and Y are Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$, then $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

2. *If X, Y and Z are Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively, then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{T}\mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

Lemma 3.6 ([2]). *Let $1 < p, q < \infty$ and let D be a domain in $\mathbb{R}^N$.*

1. *Let $m(\lambda)$ be a bounded function defined on a subset Λ in a complex plane $\mathbb{C}$ and let $M_m(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$.*

Then

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \le C_{n,q,D}\|m\|_{L_\infty}.$$

2. *Let* $n(\tau)$ *be a* C^1 *function defined on* $\mathbb{R} \setminus \{0\}$ *that satisfies the conditions:* $|n(\tau)| \le \gamma$ *and* $|\tau n'(\tau)| \le \gamma$ *with some constants* $\gamma > 0$ *for any* $\gamma \in \mathbb{R} \setminus \{0\}$. *Let* T_n *be an operator valued Fourier multiplier defined by* $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ *for any* f *with* $\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)$. *Then,* T_n *is extended to bounded linear operator from* $L_q\mathbb{R}, L_q(D)$ *into itself. Moreover, denoting this extension also by* T_n, *we have*

$$\|T_n\|_{\mathcal{L}(L_q(\mathbb{R},L_q(D)))} \le C_{p,q,D}\gamma.$$

Hereinafter, we consider problem (3.3). Let $\mathcal{A}(\lambda)$ be the operator given in Theorem 3.4, and let $\mathbf{u} = \mathcal{A}(\lambda)F_\lambda(\mathbf{g}, \mathbf{k})$ in (3.3), where $F_\lambda(\mathbf{g}, \mathbf{k}) = (\mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k})$. By Theorem 3.4, (3.3) and (3.4), we have

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u}) - B_1(\lambda)(\mathbf{u}) = \mathbf{g} - \mathcal{C}_1(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n} - B_2(\lambda)(\mathbf{u}) = \mathbf{k} - \mathcal{C}_2(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \end{cases} \tag{3.5}$$

where we have set

$$\begin{aligned} \mathcal{C}_1(\lambda)\mathbf{F} &= \lambda^{-1}\nabla(\gamma_1\gamma_3\operatorname{div}\mathcal{A}(\lambda)\mathbf{F}) + \delta_1(\lambda+\delta_2)^{-1}\operatorname{Div}(g_\alpha(\nabla\mathcal{A}(\lambda)\mathbf{F},\tau_1) + \delta_3\mathbf{D}(\mathcal{A}(\lambda)\mathbf{F})), \\ \mathcal{C}_2(\lambda)\mathbf{F} &= -(\lambda^{-1}\gamma_1\gamma_3\operatorname{div}\mathcal{A}(\lambda)\mathbf{F} + \delta_1(\lambda+\delta_2)^{-1}(g_\alpha(\nabla\mathcal{A}(\lambda)\mathbf{F},\tau_1) + \delta_3\mathbf{D}(\mathcal{A}(\lambda)\mathbf{F})))\mathbf{n}. \end{aligned} \tag{3.6}$$

Let $\mathcal{E}_\lambda\mathbf{u} = (\gamma_2\lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u}) - B_1(\lambda)(\mathbf{u}), \mathbf{S}(\mathbf{u})\mathbf{n} - B_2(\lambda)(\mathbf{u}))$ and $\mathcal{G}_\lambda\mathbf{F} = (C_1(\lambda)\mathbf{F}, C_2(\lambda)\mathbf{F})$. For $\mathbf{F} = (F_1, \mathbf{F}', \mathbf{F}_5) \in \mathcal{X}_q(\Omega)$ with $\mathbf{F}' = (\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \in \mathcal{Y}_q(\Omega)$, we may write Eq. (3.5) in the form:

$$\mathcal{E}_\lambda\mathcal{A}(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) = (\mathbf{I} - \mathcal{G}_\lambda F_\lambda)(\mathbf{g},\mathbf{k}), \tag{3.7}$$

where $\mathbf{I}$ is the identity map from $Y_q(\Omega)$ into itself.

Let λ_1 be any positive number $\ge \lambda_0$. By (1.11), Lemma 2.2(1), (2.10), Lemma 3.5, Lemma 3.6 and Theorem 3.4, we have

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{C}_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le C\lambda_1^{-1} \quad (\ell = 0,1), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\lambda^{1/2}\mathcal{C}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le C\lambda_1^{-1} \quad (\ell = 0,1), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^{N^2})}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{C}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le C\lambda_1^{-1} \quad (\ell = 0,1). \end{aligned} \tag{3.8}$$

In fact, for any $n \in \mathbb{N}$, $\lambda_j \in \Sigma_{\epsilon,\lambda_1}$, $\mathbf{F}_j \in \mathcal{Y}_q(\Omega)$, and independent, symmetric, $\{-1,1\}$-valued random variables r_j $(j = 1,\ldots,n)$, we have

$$\begin{aligned} &\int_0^1 \left\|\sum_{j=1}^n r_j(u)\nabla C_2(\lambda_j)\mathbf{F}_j\right\|_{L_q(\Omega)} du \\ &\le C_{\rho_1}\int_0^1 \left(\left\|\sum_{j=1}^n r_j(u)\lambda_j^{-1}\mathcal{A}(\lambda_j)\mathbf{F}_j\right\|_{W_q^2(\Omega)} + \left\|\sum_{j=1}^n r_j(u)(\lambda_j+\delta_2)^{-1}\mathcal{A}(\lambda_j)\mathbf{F}_j\right\|_{W_q^2(\Omega)}\right) du \\ &\le C_{\rho_1}(\lambda_1^{-1} + (\lambda_1+\delta_2)^{-1})\int_0^1 \left\|\sum_{j=1}^n r_j(u)\mathcal{A}(\lambda_j)\mathbf{F}_j\right\|_{W_q^2(\Omega)} du \\ &\le C_{\rho_1}\lambda_1^{-1}\int_0^1 \left\|\sum_{j=1}^n r_j(u)\mathbf{F}_j\right\|_{L_q(\Omega)} du. \end{aligned}$$

Analogously, we have

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) C_1(\lambda_j)\mathbf{F}_j \right\|_{L_q(\Omega)} du \le C_{\rho_1}(\lambda_1^{-1} + (\lambda_1+\delta_2)^{-1}) \int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathcal{A}(\lambda_j)\mathbf{F}_j \right\|_{W_q^2(\Omega)} du$$
$$\le C_{\rho_1}\lambda_1^{-1} \int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathbf{F}_j \right\|_{L_q(\Omega)} du.$$

And also,

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u)\lambda_j^{1/2} C_2(\lambda_j)\mathbf{F}_j \right\|_{L_q(\Omega)} du \le C_{\rho_1}(\lambda_1^{-1} + (\lambda_1+\delta_2)^{-1}) \int_0^1 \left\| \sum_{j=1}^n r_j(u)\lambda_j^{1/2}\mathcal{A}(\lambda_j)\mathbf{F}_j \right\|_{W_q^1(\Omega)} du$$
$$\le C_{\rho_1}\lambda_1^{-1} \int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathbf{F}_j \right\|_{L_q(\Omega)} du.$$

Thus, we have (3.5) for $\ell = 0$. Analogously, we have (3.5) for $\ell = 1$.

In particular, by (3.8) we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),\mathcal{Y}_q(\Omega))}(\{(\tau\partial_\tau)^\ell F_\lambda \mathcal{G}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le C\lambda_1^{-1} \quad (\ell = 0, 1). \tag{3.9}$$

We choose $\lambda_1 \ge \lambda_0$ so large that

$$C\lambda_1^{-1} \le 1/2 \tag{3.10}$$

in (3.9). Let $\|(\mathbf{g},\mathbf{k})\|_{Y_q(\Omega)} = \|\mathbf{g}\|_{L_q(\Omega)} + \|\mathbf{k}\|_{W_q^1(\Omega)}$ and $\|\mathbf{F}\|_{\mathcal{Y}_q(\Omega)} = \sum_{k=2,3,4}\|\mathbf{F}_k\|_{L_q(\Omega)}$. By (3.10)

$$\|F_\lambda[\mathcal{G}_\lambda F_\lambda(\mathbf{g},\mathbf{k})]\|_{\mathcal{Y}_q(\Omega)} = \|F_\lambda \mathcal{G}_\lambda(F_\lambda(\mathbf{g},\mathbf{k}))\|_{\mathcal{Y}_q(\Omega)} \leqq (1/2)\|F_\lambda(\mathbf{g},\mathbf{k})\|_{\mathcal{Y}_q(\Omega)}.$$

Since $\|F_\lambda(\mathbf{g},\mathbf{k})\|_{\mathcal{Y}_q(\Omega)}$ is equivalent norms to $\|(\mathbf{g},\mathbf{k})\|_{Y_q(\Omega)}$ provided that $\lambda \neq 0$, $\mathbf{I} - \mathcal{G}_\lambda F_\lambda$ has its inverse operator $(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}$ in $Y_q(\Omega)$. By (3.7), $\mathcal{E}_\lambda \mathcal{A}(\lambda)F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}(\mathbf{g},\mathbf{k}) = (\mathbf{g},\mathbf{k})$, so that problem (3.3) admits a solution $\mathbf{u} = \mathcal{A}(\lambda)F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}(\mathbf{g},\mathbf{k})$. The uniqueness follows from the existence of solutions to the dual equations. Moreover, $F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1} = (\mathbf{I} - F_\lambda \mathcal{G}_\lambda)^{-1}F_\lambda$. Thus, if we define the operator $\mathcal{B}(\lambda) = \mathcal{A}(\lambda)(\mathbf{I} - F_\lambda\mathcal{G}_\lambda)^{-1}$, then $\mathbf{u} = \mathcal{B}(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) = \mathcal{A}(\lambda)F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}(\mathbf{g},\mathbf{k})$ is a unique solution of problem (3.3), and by Theorem 3.4, Lemma 3.5, (3.9), and (3.10), we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^{\tilde{N}})}(\{(\tau\partial\tau)^\ell(G_\lambda\mathcal{B}(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) \le C \quad (\ell = 0, 1). \tag{3.11}$$

For $\mathbf{F} = (F_1, \mathbf{F}', \mathbf{F}_5) \in \mathcal{X}_q(\Omega)$ with $\mathbf{F}' = (\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \in \mathcal{Y}_q(\Omega)$, let $R(\lambda)\mathbf{F}$ be defined by

$$R(\lambda)\mathbf{F} = (\lambda^{-1}(F_1 - \gamma_1 \mathrm{div}\,\mathcal{B}(\lambda)\mathbf{F}')), \mathcal{B}(\lambda)\mathbf{F}', (\lambda+\delta_2)^{-1}(\delta_3\mathbf{D}(\mathcal{B}(\lambda)(\lambda)\mathbf{F}') + g_\alpha(\nabla\mathcal{B}(\lambda)\mathbf{F}', \tau_1) + \mathbf{F}_5),$$

and then by Lemma 3.5 and (3.11), we see that $R(\lambda)$ is the required operator in Theorem 3.2, which completes the proof of Theorem 3.2.

4. L_p–L_q maximal regularity for problem (1.10)

In this section, we shall prove the following theorem concerned with the L_p–L_q maximal regularity.

Theorem 4.1. *Let $1 < p, q < \infty$, $N < r < \infty$, and $\max(q, q') \le r$ ($q' = \frac{q}{q-1}$). Let T be any positive number. Assume that Ω is a uniform $W_r^{2-\frac{1}{r}}$ domain. Let*

$$\rho_0 \in W_q^1(\Omega), \qquad \mathbf{u}_0 \in B_{q,p}^{2(1-1/p)}(\Omega)^N, \qquad \tau_0 \in W_q^1(\Omega)^{N^2}$$

be initial data for problem (1.10), and let

$$f \in L_p((0,T), W^1_q(\Omega)), \qquad \mathbf{g} \in L_p((0,T), L_q(\Omega)), \qquad \mathbf{h} \in L_p((0,T), W^1_q(\Omega)^{N^2}),$$
$$\mathbf{k} \in L_p((0,T), W^1_q(\Omega)^N) \cap W^1_p((0,T), \mathbf{W}^{-1}_q(\Omega)^N),$$

be right members for problem (1.10). Assume that they satisfy the compatibility condition:

$$(\mathbf{T}(\mathbf{u}_0, \gamma_3\rho_0) + \delta_1\tau_0)\mathbf{n} = \mathbf{k}|_{t=0} \quad on\ \Gamma_1, \qquad \mathbf{u}_0 = 0 \quad on\ \Gamma_0. \tag{4.1}$$

Then, problem (1.10) admits unique solutions ρ, $\mathbf{u}$ and τ with

$$\rho \in W^1_p((0,T), W^1_q(\Omega)), \qquad \mathbf{u} \in L_p((0,T), W^2_q(\Omega)^N) \cap W^1_p((0,T), L_q(\Omega)^N), \qquad \tau \in W^1_p((0,T), W^1_q(\Omega)^{N^2})$$

possessing the estimate:

$$\begin{aligned}[\![(\rho, \mathbf{u}, \tau)]\!]_t \le Ce^{\gamma t}(&\|(\rho_0, \tau_0)\|_{W^1_q(\Omega)} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|(f, \mathbf{g}, \mathbf{h})\|_{L_p((0,t), W^{1,0}_q(\Omega))} \\ &+ \|\mathbf{k}\|_{L_p((0,t), W^1_q(\Omega))} + \|\partial_t\mathbf{k}\|_{L_p((0,t), \mathbf{W}^{-1}_q(\Omega))})\end{aligned} \tag{4.2}$$

for any $t \in (0,T)$ with some positive constants γ and C, where we have set

$$[\![(\rho, \mathbf{u}, \tau)]\!]_t = \|(\rho, \tau)\|_{W^1_p((0,t), W^1_q(\Omega))} + \|\mathbf{u}\|_{L_p((0,t), W^2_q(\Omega))} + \|\partial_t\mathbf{u}\|_{L_p((0,t), L_q(\Omega))} \tag{4.3}$$

and the constant C in (4.2) depends on ρ_0 and ρ_1.

To prove Theorem 4.1, first of all we transform problem (1.10) to the zero initial data case. To this end, we take a domain Ω_1 such that $\partial\Omega_1 = \Gamma_0$ and $\Omega \subset \Omega_1$. The Ω_1 is a uniform $W^{2-1/r}_r$ $(N < r < \infty)$ domain. Let $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)$ be an initial velocity field for problem (1.10) and let $\tilde{\mathbf{u}}_0 = (\tilde{u}_{01}, \ldots, \tilde{u}_{0N})$ be an extension of $\mathbf{u}_0$ to Ω_1 such that $\mathbf{u}_0 = \tilde{\mathbf{u}}_0$ on Ω and $\|\tilde{\mathbf{u}}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega_1)} \le C\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}$. We consider the time-shifted heat equations:

$$\partial_t v_j + \lambda_0 v_j - \mu\Delta v_j = 0 \quad \text{in } \Omega_1 \times (0,\infty), \qquad v_j|_{\Gamma_0} = 0, \qquad v_j|_{t=0} = \tilde{u}_{0j} \tag{4.4}$$

$(j = 1, \ldots, N)$. Since $\tilde{u}_{0j}$ satisfies the compatibility condition: $\tilde{u}_{0j}|_{\Gamma_0} = u_{0j}|_{\Gamma_0} = 0$ as follows from (4.1), employing the similar argumentation to that in Shibata [21,22], we see that there exist v_j $(j = 1, \ldots, N)$ such that

$$\begin{aligned}&v_j \in L_p((0,\infty), W^2_q(\Omega_1)) \cap W^1_p((0,\infty), L_q(\Omega_1)), \\ &\|\partial_t v_j\|_{L_p((0,\infty), L_q(\Omega_1))} + \|v_j\|_{L_p((0,\infty), W^2_q(\Omega_1))} \le C\|\tilde{u}_{0j}\|_{B^{2(1-1/p)}_{q,p}(\Omega_1)} \le C\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}.\end{aligned} \tag{4.5}$$

Set $\mathbf{v} = (v_1, \ldots, v_N)$. In problem (1.10), we set $\rho = \rho_0 + \theta$, $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\tau = \tau_0 + \omega$, and then θ, $\mathbf{w}$ and ω satisfy the following equations:

$$\begin{cases} \partial_t\theta + \gamma_1 \operatorname{div}\mathbf{w} = f' & \text{in } \Omega \times (0,T), \\ \gamma_2\partial_t\mathbf{w} - \operatorname{Div}\mathbf{T}(\mathbf{w}, \gamma_3\theta) = \delta_1 \operatorname{Div}\omega + \mathbf{g}' & \text{in } \Omega \times (0,T), \\ \partial_t\omega + \delta_2\omega - g_\alpha(\nabla\mathbf{w}, \tau_1) = \delta_3\mathbf{D}(\mathbf{w}) + \mathbf{h}' & \text{in } \Omega \times (0,T), \\ (\mathbf{T}(\mathbf{w}, \gamma_3\theta) + \delta_1\omega)\mathbf{n} = \mathbf{k}' & \text{on } \Gamma_1 \times (0,T), \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \times (0,T), \\ (\theta, \mathbf{w}, \omega)|_{t=0} = (0,0,0) & \text{in } \Omega, \end{cases} \tag{4.6}$$

with $f' = f - \gamma_1 \operatorname{div}\mathbf{v}$, $\mathbf{g}' = \mathbf{g} - \gamma_2\partial_t\mathbf{v} + \operatorname{Div}\mathbf{T}(\mathbf{v}, \gamma_3\rho_0) + \delta_1 \operatorname{Div}\tau_0$, $\mathbf{h}' = \mathbf{h} - \delta_2\tau_0 + g_\alpha(\nabla\mathbf{v}, \tau_1) + \delta_3\mathbf{D}(\mathbf{v})$, and $\mathbf{k}' = \mathbf{k} - (\mathbf{T}(\mathbf{v}, \gamma_3\rho_0) + \delta_1\tau_0)\mathbf{n}$. By (4.5) and Lemma 2.2(1) with $s = r$, (2.10) and (1.11), we have

$$\|(f', \mathbf{g}', \mathbf{h}')\|_{L_p((0,t), W^{1,0}_q(\Omega))} + \|\mathbf{k}'\|_{L_p((0,t), W^1_q(\Omega))} + \|\partial_t\mathbf{k}'\|_{L_p((0,t), \mathbf{W}^{-1}_q(\Omega))} \le C\mathcal{D}_t \tag{4.7}$$

with

$$\begin{aligned}\mathcal{D}_t = \|(\rho_0, \tau_0)\|_{W^1_q(\Omega)} &+ \|\mathbf{u}_0\|_{B^{2-1/p}_{q,p}(\Omega)} + \|(f, \mathbf{g}, \mathbf{h})\|_{L_p((0,t), W^{1,0}_q(\Omega))} \\ &+ \|\mathbf{k}\|_{L_p((0,t), W^1_q(\Omega))} + \|\partial_t\mathbf{k}\|_{L_p((0,t), \mathbf{W}^{-1}_q(\Omega))}.\end{aligned}$$

Thus, from now on we consider problem (4.6). We modify the right members to consider the problem on $\mathbb{R}$ for time. Given any function $f(\cdot,t)$ defined on $(0,T)$, let f_0 denote the zero extension of f to $(-\infty,0)$, namely $f_0(\cdot,t)=f(\cdot,t)$ for $t\in(0,T)$ and $f_0(\cdot,t)=0$ for $t\in(-\infty,0)$. Let E_t be an operator defined by

$$[E_t f](\cdot,s)=\begin{cases} f_0(\cdot,s) & \text{for } s<t,\\ f_0(\cdot,2t-s) & \text{for } s>t.\end{cases} \tag{4.8}$$

Obviously, $[E_t f](\cdot,s)=0$ for $s\notin(0,2t)$. Moreover, if $f|_{t=0}=0$, then we have

$$\partial_s[E_t f](\cdot,s)=\begin{cases} 0 & \text{for } s\notin(0,2t),\\ (\partial_s f)(\cdot,s) & \text{for } s\in(0,t),\\ -(\partial_s f)(\cdot,2t-s) & \text{for } s\in(t,2t).\end{cases} \tag{4.9}$$

For $t\in(0,T)$, let

$$F=E_t[f'],\qquad \mathbf{G}=E_t[\mathbf{g}'],\qquad \mathbf{H}=E_t[\mathbf{h}'],\qquad \mathbf{K}=E_t[\mathbf{k}'].$$

By the compatibility condition (4.1), $\mathbf{k}'|_{t=0}=0$, so that by (4.9), we have

$$\begin{aligned} &\partial_s\mathbf{K}=(\partial_s\mathbf{k}')(\cdot,s)\quad \text{for } s\in(0,t),\qquad \partial_s\mathbf{K}=-(\partial_s\mathbf{k}')(\cdot,2t-s)\quad \text{for } s\in(t,2t),\\ &\partial_s\mathbf{K}=0\quad \text{for } s\notin(0,2t).\end{aligned} \tag{4.10}$$

First, we consider the whole time problem:

$$\begin{cases} \partial_t\theta+\gamma_1\operatorname{div}\mathbf{w} = F & \text{in } \Omega\times\mathbb{R}\\ \gamma_2\partial_t\mathbf{w}-\operatorname{Div}\mathbf{T}(\mathbf{w},\gamma_3\theta) = \delta_1\operatorname{Div}\omega+\mathbf{G} & \text{in } \Omega\times\mathbb{R}\\ \partial_t\omega+\delta_2\omega-g_\alpha(\nabla\mathbf{w},\tau_1) = \delta_3\mathbf{D}(\mathbf{w})+\mathbf{H} & \text{in } \Omega\times\mathbb{R}\\ (\mathbf{T}(\mathbf{w},\gamma_3\theta)+\delta_1\omega)\mathbf{n} = \mathbf{K} & \text{on } \Gamma_1\times\mathbb{R},\\ \mathbf{w} = 0 & \text{on } \Gamma_0\times\mathbb{R}.\end{cases} \tag{4.11}$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace–Fourier transform and the inverse Laplace–Fourier transform with respect to t defined by

$$\mathcal{L}[f](\lambda)=\hat f=\int_{-\infty}^{\infty}e^{-(\gamma+i\tau)t}f(t)\,dt\quad(\lambda=\gamma+i\tau),\qquad \mathcal{L}^{-1}[g](t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(\gamma+i\tau)t}g(\tau)\,d\tau.$$

Let $\mathcal{F}_t$ and $\mathcal{F}_\tau^{-1}$ be the Fourier transform with respect to t and the inverse Fourier transform with respect to τ defined by

$$\mathcal{F}[f](\tau)=\int_{-\infty}^{\infty}e^{-i\tau t}f(t)\,dt,\qquad \mathcal{F}^{-1}[g](t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau t}g(\tau)\,d\tau.$$

We see that

$$\mathcal{L}[f](\lambda)=\mathcal{F}_t[e^{-\gamma t}f(t)],\qquad \mathcal{L}^{-1}[g](t)=e^{\gamma t}\mathcal{F}_\tau^{-1}[g(\tau)](t). \tag{4.12}$$

Applying the Laplace–Fourier transform to (4.11), we have

$$\begin{cases} \lambda\hat\theta+\gamma_1\operatorname{div}\hat{\mathbf{w}} = \hat F & \text{in } \Omega\\ \gamma_2\lambda\hat{\mathbf{w}}-\operatorname{Div}\mathbf{T}(\hat{\mathbf{w}},\gamma_3\hat\theta) = \delta_1\operatorname{Div}\hat\omega+\hat{\mathbf{G}} & \text{in } \Omega\\ \lambda\hat\omega+\delta_2\hat\omega-g_\alpha(\nabla\hat{\mathbf{w}},\tau_1) = \delta_3\mathbf{D}(\hat{\mathbf{w}})+\hat{\mathbf{H}} & \text{in } \Omega\\ (\mathbf{T}(\hat{\mathbf{w}},\gamma_3\hat\theta)+\delta_1\hat\omega)\mathbf{n} = \hat{\mathbf{K}} & \text{on } \Gamma_1,\\ \hat{\mathbf{w}} = 0 & \text{on } \Gamma_0.\end{cases} \tag{4.13}$$

Let $R(\lambda)$ be the solution operator to problem (1.12) given in Theorem 3.2, and then we have

$$(\theta,\mathbf{w},\omega)=\mathcal{L}^{-1}[R(\lambda)(\hat F,\hat{\mathbf{G}},\lambda^{1/2}\hat{\mathbf{K}},\nabla\hat{\mathbf{K}},\hat{\mathbf{H}})]. \tag{4.14}$$

Let $\Lambda_\gamma^{1/2} f$ be the operator defined by

$$\Lambda_\gamma^{1/2} f = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[f](\lambda)].$$

Note that $\lambda^{1/2}\hat{\mathbf{K}} = \mathcal{L}[\Lambda_\gamma^{1/2}\mathbf{K}]$. To estimate $(\theta, \mathbf{w}, \omega)$, we quote the Weis operator valued Fourier multiplier theorem. Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact support and the Schwartz space of rapidly decreasing X valued function, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]] \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)). \tag{4.15}$$

The following theorem is obtained by Weis [33].

Theorem 4.2. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}\left(\left\{\left(\tau\frac{d}{d\tau}\right)^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\right\}\right) \le \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (4.15) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 4.3. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

Applying the Weis theorem stated above to $(\theta, \mathbf{w}, \omega)$ defined in (4.14), we have

$$\begin{aligned}
&\|e^{-\gamma s}(\partial_t\theta, \partial_t\omega)\|_{L_p(\mathbb{R},W_q^1(\Omega))} + \|e^{-\gamma s}(\partial_t\mathbf{w}, \Lambda_\gamma^{1/2}\nabla\mathbf{w}, \nabla^2\mathbf{w})\|_{L_p(\mathbb{R},L_q(\Omega))} \\
&\quad \le C(\|e^{-\gamma s}(F, \mathbf{G}, \mathbf{H})\|_{L_p(\mathbb{R},W_q^{1,0}(\Omega))} + \|e^{-\gamma s}(\Lambda_\gamma^{1/2}\mathbf{K}, \nabla\mathbf{K})\|_{L_p(\mathbb{R},L_q(\Omega))})
\end{aligned} \tag{4.16}$$

for any $\gamma \ge \lambda_0 + 1$ with some constants C independent of γ, where λ_0 is the constant given in Theorem 3.2. By using the fact due to Shibata [23, Appendix], Lemmas 2.2 and 3.6, we can prove easily that

$$\begin{aligned}
&\|e^{-\gamma s}\Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R},L_q(\Omega))} \le C\{\|e^{-\gamma s}\partial_s f\|_{L_p(\mathbb{R},\mathbf{W}_q^{-1}(\Omega))} + \|e^{-\gamma s} f\|_{L_p(\mathbb{R},W_q^1(\Omega))}\}, \\
&\|e^{-\gamma s}\gamma f\|_{L_p(\mathbb{R},L_q(\Omega))} \le C\|e^{-\gamma s}\partial_s f\|_{L_p(\mathbb{R},L_q(\Omega))}, \\
&\|e^{-\gamma s}\partial_t f\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma s} f\|_{L_p(\mathbb{R},W_q^2(\Omega))} \le C\|e^{-\gamma s}(\partial_s f, \Lambda_\gamma^{1/2}\nabla f, \nabla^2 f)\|_{L_p(\mathbb{R},L_q(\Omega))},
\end{aligned} \tag{4.17}$$

which, combined with (4.16), furnishes that

$$\begin{aligned}
&\gamma\|e^{-\gamma s}(\theta, \mathbf{w}, \omega)\|_{L_p(\mathbb{R},W_q^{1,0}(\Omega))} + \|e^{-\gamma s}(\partial_t\theta, \partial_t\omega)\|_{L_p(\mathbb{R},W_q^1(\Omega))} + \|e^{-\gamma s}\partial_t\mathbf{w}\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma s}\mathbf{w}\|_{L_p(\mathbb{R},W_q^2(\Omega))} \\
&\quad \le C(\|e^{-\gamma s}(F, \mathbf{G}, \mathbf{H})\|_{L_p(\mathbb{R},W_q^{1,0}(\Omega))} + \|e^{-\gamma s}\nabla\mathbf{K}\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma s}\partial_s\mathbf{K}\|_{L_p(\mathbb{R},\mathbf{W}_q^{-1}(\Omega))})
\end{aligned} \tag{4.18}$$

for any $\gamma \ge \lambda_0 + 1$. By (4.8) and (4.10), we have

$$\|e^{-\gamma s}(F, \mathbf{G}, \mathbf{H})\|_{L_p(\mathbb{R},W_q^{1,0}(\Omega))} + \|e^{-\gamma s}\nabla\mathbf{K}\|_{L_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma s}\partial_s\mathbf{K}\|_{L_p(\mathbb{R},\mathbf{W}_q^{-1}(\Omega))} \le C\mathcal{D}_t \tag{4.19}$$

with some constant C independent of t. By (4.18) and (4.19), we see that

$$(\theta, \mathbf{w}, \omega)(\cdot, s) = 0 \quad \text{for } s < 0. \tag{4.20}$$

In fact, we observe that

$$\|(\theta,\mathbf{w},\omega)\|_{L_p((-\infty,0),W^{1,0}_q(\Omega))} \le \|e^{-\gamma s}(\theta,\mathbf{w},\omega)\|_{L_p(\mathbb{R},W^{1,0}_q(\Omega))} \le \gamma^{-1}\mathcal{D}_t$$

for any $\gamma \ge \lambda_0+1$, so that we have (4.20) as $\gamma\to\infty$. Combining (4.18)–(4.20), we have

$$[\![(\theta,\mathbf{w},\omega)]\!]_t \le Ce^{\gamma t}\mathcal{D}_t \tag{4.21}$$

for any $\gamma \ge \lambda_0+1$ with some constant C independent of γ. Moreover, since $[E_t f](\cdot s) = f(\cdot,s)$ for $s\in(0,t)$, by (4.11) and (4.20), the $(\theta,\mathbf{w},\omega)$ is a solution to the equations:

$$\begin{cases} \partial_s\theta + \gamma_1 \operatorname{div}\mathbf{w} = f' & \text{in } \Omega\times(0,t)\\ \gamma_2\partial_s\mathbf{w} - \operatorname{Div}\mathbf{T}(\mathbf{w},\gamma_3\theta) = \delta_1\operatorname{Div}\omega + \mathbf{g}' & \text{in } \Omega\times(0,t)\\ \partial_s\omega + \delta_2\omega - g_\alpha(\nabla\mathbf{w},\tau_1) = \delta_3\mathbf{D}(\mathbf{w}) + \mathbf{h}' & \text{in } \Omega\times(0,t)\\ (\mathbf{T}(\mathbf{w},\gamma_3\theta)+\delta_1\omega)\mathbf{n} = \mathbf{k}' & \text{on } \Gamma_1\times(0,t),\\ \mathbf{w} = 0 & \text{on } \Gamma_0\times(0,t),\\ (\theta,\mathbf{w},\omega)|_{s=0} = (0,0,0) & \text{in } \Omega. \end{cases} \tag{4.22}$$

For $0<t_1<t_2\le T$, let θ^{t_i}, $\mathbf{w}^{t_i}$, and ω^{t_i} be solutions of Eqs. (4.22) with $t=t_i$. By the uniqueness of solutions which follows from the solvability of the dual problem (cf. [25]), we have $(\theta^{t_1},\mathbf{w}^{t_1},\omega^{t_1}) = (\theta^{t_2},\mathbf{w}^{t_2},\omega^{t_2})$ for $s\in(0,t_1)$, so that if we set $(\theta,\mathbf{w},\omega)=(\theta^T,\mathbf{w}^T,\omega^T)$, then we have $(\theta,\mathbf{w},\omega)=(\theta^t,\mathbf{w}^t,\omega^t)$ for any $t\in(0,T]$. This completes the proof of Theorem 4.1.

5. A proof of the local wellposedness

In this section, we prove Theorem 1.2 by using the Banach fixed point theorem. In the sequel, we assume that $2<p<\infty$, $N<q<\infty$, and that Ω is a uniform $W^{2-1/q}_q$ domain in $\mathbb{R}^N$ $(N\ge 2)$. Let T and L be any positive numbers and let $\mathcal{I}_{L,T}$ be the space defined by

$$\begin{aligned} \mathcal{I}_{L,T} = \{(\theta,\mathbf{v},\tau) \mid \theta\in W^1_p((0,T),W^1_q(\Omega)),\quad \mathbf{v}\in W^1_p((0,T),L_q(\Omega))\cap L_p((0,T),W^2_q(\Omega)),\\ \tau\in W^1_p((0,T),W^1_q(\Omega)),\ (\theta,\mathbf{v},\tau)|_{t=0} = (0,\mathbf{u}_0,0) \text{ in } \Omega,\ [\![(\theta,\mathbf{v},\tau)]\!]_T \le L\}. \end{aligned} \tag{5.1}$$

Since we choose $T>0$ small enough and $L>0$ large enough eventually, we may assume that $0<T\le 1$ and $L\ge 1$. Moreover, we choose ρ_1 in (1.11) in such a way that $\|\tau_0\|_{W^1_q(\Omega)} \le R \le \rho_1$. Given $(\kappa,\mathbf{w},\varphi)\in\mathcal{I}_{L,T}$, let $\theta,\mathbf{v}$ and ψ be solutions to problem:

$$\begin{cases} \theta_t + (\rho_*+\theta_0)\operatorname{div}\mathbf{v} = F(\kappa,\mathbf{w}) & \text{in } \Omega\times(0,T),\\ (\rho_*+\theta_0)\mathbf{v}_t - \operatorname{Div}S(\mathbf{v}) + \nabla(P'(\rho_*+\theta_0)\theta) = \beta\operatorname{Div}\psi + \mathbf{g} + \mathbf{G}(\mathbf{w},\kappa,\varphi) & \text{in } \Omega\times(0,T),\\ \psi_t + \gamma\psi - g_\alpha(\nabla\mathbf{u},\tau_0) - \delta\mathbf{D}(\mathbf{v}) = -\gamma\tau_0 + \mathbf{L}(\mathbf{w},\varphi) & \text{in } \Omega\times(0,T),\\ (\mathbf{S}(\mathbf{v}) - P'(\rho_*+\theta_0)\theta\mathbf{I} + \beta\psi)\mathbf{n} = \mathbf{h} + \mathbf{H}(\mathbf{w},\kappa,\varphi) & \text{on } \Gamma_1\times(0,T),\\ \mathbf{v} = 0 & \text{on } \Gamma_0\times(0,T),\\ (\theta,\mathbf{v},\tau)|_{t=0} = (0,\mathbf{u}_0,0) & \text{in } \Omega. \end{cases} \tag{5.2}$$

In the sequel, C denotes generic constants independent of R and L, and C_R denotes generic constants independent of L. M_i denotes some special constants. The values of C and C_R may change from line to line. First, we estimate the right-hand side of (1.8). By the Sobolev inequality (cf. Lemma 2.2(1)), Hölder inequality and the identities: $\kappa(\cdot,t) = \int_0^t \partial_s\kappa(\cdot,s)\,ds$ and $\varphi(\cdot,t)=\int_0^t\partial_s\varphi(\cdot,s)\,ds$, we have

$$\begin{aligned} &\sup_{t\in(0,T)}\left\|\int_0^t \nabla\mathbf{w}(\cdot,s)ds\right\|_{L_\infty(\Omega)} \le M_1T^{1/p'}L, && \sup_{t\in(0,T)}\left\|\int_0^t \nabla\mathbf{w}(\cdot,s)ds\right\|_{W^1_q(\Omega)} \le M_1T^{1/p'}L\\ &\sup_{t\in(0,T)}\|\kappa(\cdot,s)\|_{L_\infty(\Omega)} \le M_1T^{1/p'}L, && \sup_{t\in(0,T)}\|\kappa(\cdot,s)\|_{W^1_q(\Omega)} \le M_1T^{1/p'}L\\ &\sup_{t\in(0,T)}\|\varphi(\cdot,s)\|_{L_\infty(\Omega)} \le M_1T^{1/p'}L, && \sup_{t\in(0,T)}\|\varphi(\cdot,s)\|_{W^1_q(\Omega)} \le M_1T^{1/p'}L \end{aligned} \tag{5.3}$$

with $p' = p/(p-1)$. To determine functions with respect to κ, φ and $\int_0^t \nabla \mathbf{w}(\cdot, s)\,ds$, in view of the range condition: $\frac{\rho_*}{2} < \rho_* + \theta_0 < 2\rho_*$ in Theorem 1.2 and (1.6), we choose T small enough in such a way that $M_1 T^{1/p'} L \le \rho_*/2$, $M_1 T^{1/p'} L \le \sigma$ and $M_1 T^{1/p'} L \le 1$, and then we have

$$\frac{\rho_*}{4} < \rho_* + \theta_0 + \ell\kappa < 4\rho_* \quad (\ell \in [0,1]), \qquad \sup_{t\in(0,T)} \left\| \int_0^t \nabla \mathbf{w}(\cdot, s)\,ds \right\|_{L_\infty(\Omega)} < \sigma. \tag{5.4}$$

Recall that $\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-\frac{1}{p})}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \le R$ (cf. Theorem 1.2 (1.14)). By (5.3) and (5.4) we have

$$\sup_{t\in(0,T)} \left\| V_i \left(\int_0^t \nabla \mathbf{w}(\cdot, s)\,ds \right) \right\|_{L_\infty(\Omega)} \le C T^{1/p'} L, \qquad \sup_{t\in(0,T)} \left\| \nabla \mathbf{W} \left(\int_0^t \nabla \mathbf{w}(\cdot, s) ds \right) \right\|_{W_q^1(\Omega)} \le C T^{1/p'} L$$
$$\sup_{t\in(0,T)} \left\| \nabla \int_0^1 P''(\rho_* + \theta_0 + \ell\kappa)(1-\ell) d\ell \right\|_{L_q(\Omega)} \le C(R + T^{1/p'} L) \tag{5.5}$$

where $i = D$, div , W, and $\mathbf{W} = \mathbf{W}(\mathbf{K})$ is any matrix of functions with respect to $\mathbf{K}$. By Lemma 2.2(1), (2.10), (5.3), (5.4) and (5.5), we have

$$\begin{aligned}
&\|(0, \mathbf{g}, -\gamma\tau_0)\|_{L_p((0,T),W_q^{1,0}(\Omega))} + \|\mathbf{h}\|_{L_p((0,T),W_q^1(\Omega))} + \|\partial_t \mathbf{h}\|_{L_p((0,T),\mathbf{W}_q^{-1}(\Omega))} \le CRT^{1/p},\\
&\|(F(\kappa, \mathbf{w}), \mathbf{G}(\mathbf{w}, \kappa, \varphi), \mathbf{L}(\mathbf{w}, \varphi))\|_{L_p((0,T),W_q^{1,0}(\Omega))} \le C(L+R)^2 (T^{1/p'} + T^{1/p}),\\
&\|\mathbf{H}(\mathbf{w}, \kappa, \varphi)\|_{L_p((0,T),W_q^1(\Omega))} \le C(L+R)^2 (T^{1/p'} + T^{1/p}),
\end{aligned} \tag{5.6}$$

where we have used the fact that $\partial_t \mathbf{h} = 0$.

To obtain the following estimates,

$$\sup_{t\in(0,T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le C(\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W_q^2(\Omega))} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}) \tag{5.7}$$

we use the embedding relation:

$$L_p((0,\infty), X_1) \cap W_p^1((0,\infty), X_0) \subset BUC((0,\infty), [X_0, X_1]_{1-1/p,p}) \tag{5.8}$$

for any two Banach spaces X_0 and X_1 such that X_1 dense in X_0 and $1 < p < \infty$ (cf. [1]). In fact, as was seen in Section 4, let $\tilde{\mathbf{u}}_0 \in B_{q,p}^{2(1-1/p)}(\Omega_1)$ be an extension of $\mathbf{u}_0$ such that $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ on Ω and $\|\tilde{\mathbf{u}}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega_1)} \le C\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$, and then there exists a $\mathbf{Z} \in W_p^1((0,\infty), L_q(\Omega)^N) \cap L_p((0,\infty), W_q^2(\Omega)^N)$ which satisfies the equations:

$$\partial_t \mathbf{Z} + \lambda_0 \mathbf{Z} - \mu\Delta\mathbf{Z} = 0 \quad \text{in } \Omega_1 \times (0,\infty), \qquad \mathbf{Z}|_{\Gamma_0} = 0, \qquad \mathbf{Z}|_{t=0} = \tilde{\mathbf{u}}_0 \quad \text{in } \Omega_1,$$

and possesses the estimate:

$$\|\partial_t \mathbf{Z}\|_{L_p((0,\infty),L_q(\Omega_1))} + \|\mathbf{Z}\|_{L_p((0,\infty),W_q^2(\Omega_1))} \le C\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \tag{5.9}$$

with some constant C. Let $\mathbf{z} = \mathbf{w} - \mathbf{Z}$. Since $\mathbf{z}|_{t=0} = 0$, by (4.8) and (4.9) we have

$$\begin{aligned}
&\|E_T \mathbf{z}\|_{W_p^1((0,\infty),L_q(\Omega))} + \|E_T \mathbf{z}\|_{L_p((0,\infty),W_q^2(\Omega))} \le C\{\|\mathbf{z}\|_{W_p^1((0,T),L_q(\Omega))} + \|\mathbf{z}\|_{L_p((0,T),W_q^2(\Omega))}\}\\
&\quad \le C\{\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W_q^2(\Omega))} + \|\partial_t \mathbf{Z}\|_{L_p((0,\infty),L_q(\Omega))} + \|\mathbf{Z}\|_{L_p((0,\infty),W_q^2(\Omega))}\}.
\end{aligned} \tag{5.10}$$

Thus, noting that $\mathbf{w} = \mathbf{Z} + E_T \mathbf{z}$ for $t \in (0,T)$ and using (5.8), we have

$$\begin{aligned}
&\sup_{t\in(0,T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le \sup_{t\in(0,\infty)} \|\mathbf{Z}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \sup_{t\in(0,\infty)} \|E_T \mathbf{z}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)}\\
&\quad \le C\{\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W_q^2(\Omega))} + \|\partial_t \mathbf{Z}\|_{L_p(\mathbb{R}_+,L_q(\Omega))} + \|\mathbf{Z}\|_{L_p(\mathbb{R}_+,W_q^2(\Omega))}\},
\end{aligned} \tag{5.11}$$

which, combined with (5.9), furnishes (5.7). Since $B^{2(1-1/p)}_{q,p}(\Omega) \subset W^1_q(\Omega)$ as follows from the assumption: $2 < p < \infty$ by (5.7) and the fact: $\sup_{t\in(0,T)} \| \int_0^t \nabla\mathbf{w}(\cdot, s)ds\|_{L_\infty(\Omega)} \le 1$, we have

$$\sup_{t\in(0,T)} \|\mathbf{w}(\cdot,t)\|_{W^1_q(\Omega)} \le \sup_{t\in(0,T)} \|\mathbf{w}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}$$
$$\le C\{\|\mathbf{w}_t\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W^2_q(\Omega))} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\}.$$

By (5.1) we have

$$\sup_{t\in(0,T)} \|\mathbf{w}(\cdot,t)\|_{W^1_q(\Omega)} \le C(L+R). \tag{5.12}$$

Writing $V'_j(\mathbf{K}) = \partial V_j/\partial\mathbf{K}$ for $j = D$ and $= \mathrm{div}$, we have

$$\begin{aligned}
\partial_t\mathbf{H}(\mathbf{w},\kappa,\varphi) = &- \left\{\mu V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\partial_t\mathbf{w} + \mu\left(V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w}\right.\\
&\left.+(\nu-\mu)\left(V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\partial_t\nabla\mathbf{w} + \left(V'_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w}\right)\mathbf{I}\right\}\mathbf{n}\\
&-\left\{\mu(\mathbf{D}(\partial_t\mathbf{w}) + V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\partial_t\mathbf{w} + \left(V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w})\right\}V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n}\\
&-\left\{\mu\left\{\left(\mathbf{D}(\mathbf{w}) + V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\right\}V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right\}\mathbf{n}\\
&-(\nu-\mu)\left\{(\mathrm{div}\,(\partial_t\mathbf{w}) + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\partial_t\mathbf{w} + \left(V'_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w})\mathbf{I}\right\}V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n}\\
&-\left\{(\nu-\mu)\left\{\left(\mathrm{div}\,\mathbf{w} + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\mathbf{I}\right\}V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right\}\mathbf{n}\\
&+\left(2\int_0^1 P''(\rho_*+\theta_0+\ell\kappa)(1-\ell)\,d\ell\,\kappa\partial_t\kappa + \int_0^1 P'''(\rho_*+\theta_0+\ell\kappa)(1-\ell)\ell\,d\ell\,\kappa^2\partial_t\kappa\right)\mathbf{n}\\
&+\left\{(P(\rho_*+\theta_0+\kappa) - P(\rho_*))V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w} + P'(\rho_*+\theta_0+\kappa)\partial_t\kappa V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\right\}\mathbf{n}\\
&-\left\{\beta\partial_t\varphi V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right) + \beta(\varphi+\tau_0)V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right\}\mathbf{n}.
\end{aligned}$$

Applying Lemma 2.2 and using (5.12), (5.3) and (5.4), we have

$$\|\partial_t\mathbf{H}(\mathbf{w},\kappa,\varphi)\|_{L_p((0,T),\mathbf{W}^{-1}_q(\Omega))} \le C(L+R)^2(T^{1/p'} + T^{1/p}). \tag{5.13}$$

Thus, applying Theorem 4.1 to problem (5.2) and using (5.6) and (5.13), we have

$$[\![(\theta,\mathbf{v},\psi)]\!]_T \le C_R(L+R)^2(T^{1/p'} + T^{1/p}). \tag{5.14}$$

Choosing $T > 0$ so small that $C_R(L+R)^2(T^{1/p'} + T^{1/p}) \le L$ in (5.14), we have

$$[\![(\theta,\mathbf{v},\psi)]\!]_T \le L. \tag{5.15}$$

Let us define a map Φ by $\Phi(\kappa,\mathbf{w},\varphi) = (\theta,\mathbf{v},\psi)$, and then by (5.15) Φ is a map from $\mathcal{I}_{L,T}$ into itself. For $(\kappa_i,\mathbf{w}_i,\varphi_i) \in \mathcal{T}_{L,T}$ $(i = 1,2)$ let $(\theta,\mathbf{v},\psi) = \Phi(\kappa_1,\mathbf{w}_1,\varphi_1) - \Phi(\kappa_2,\mathbf{w}_2,\varphi_2)$, and let

$$\mathcal{F} = F(\kappa_1,\mathbf{w}_1) - F(\kappa_2,\mathbf{w}_2), \quad \mathcal{G} = \mathbf{G}(\mathbf{w}_1,\kappa_1,\varphi_1) - \mathbf{G}(\mathbf{w}_2,\kappa_2,\varphi_2),$$
$$\mathcal{L} = \mathbf{L}(\mathbf{w}_1,\varphi_1) - \mathbf{L}(\mathbf{w}_2,\varphi_2), \quad \mathcal{H} = \mathbf{H}(\mathbf{w}_1,\kappa_1,\varphi_1) - \mathbf{H}(\mathbf{w}_2,\kappa_2,\varphi_2),$$

then by (5.2) we have

$$\begin{cases} \theta_t + (\rho_* + \theta_0)\operatorname{div} \mathbf{v} = \mathcal{F} & \text{in } \Omega \times (0,T), \\ (\rho_* + \theta_0)\mathbf{v}_t - \operatorname{Div} S(\mathbf{v}) + \nabla(P'(\rho_* + \theta_0)\theta) = \beta \operatorname{Div} \psi + \mathcal{G} & \text{in } \Omega \times (0,T), \\ \psi_t + \gamma\psi - g_\alpha(\nabla \mathbf{u}, \tau_0) - \delta \mathbf{D}(\mathbf{v}) = \mathcal{L} & \text{in } \Omega \times (0,T), \\ (\mathbf{S}(\mathbf{v}) - P'(\rho_* + \theta_0)\theta \mathbf{I} + \beta\psi)\mathbf{n} = \mathcal{H} & \text{on } \Gamma_1 \times (0,T), \\ \mathbf{v} = 0 & \text{on } \Gamma_0 \times (0,T), \\ (\theta, \mathbf{v}, \tau)|_{t=0} = (0,0,0) & \text{in } \Omega. \end{cases} \tag{5.16}$$

Since

$$\sup_{0 \in (0,T)} \|(\mathbf{v}_1 - \mathbf{v}_2)(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq C(\|\partial_t(\mathbf{v}_1 - \mathbf{v}_2)\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L_p((0,T),W^2_q(\Omega))})$$

as follows from (5.7), employing the same argumentation as in proving (5.6) and (5.13) we have

$$\begin{aligned} &\|(\mathcal{F}, \mathcal{G}, \mathcal{L})\|_{L_p((0,T),W^{1,0}_q(\Omega))} + \|\mathcal{K}\|_{L_p((0,T),W^1_q(\Omega))} + \|\partial_t \mathcal{K}\|_{L_p((0,T),\mathbf{W}^{-1}_q(\Omega))} \\ &\quad \leq C(R+L)(T^{1/p'} + T^{1/p})[\![(\kappa_1, \mathbf{w}_1, \varphi_1) - (\kappa_2, \mathbf{w}_2, \varphi_2)]\!]_T. \end{aligned}$$

Thus, applying Theorem 4.1 to Eqs. (5.16), we have

$$[\![\Phi(\kappa_1, \mathbf{w}_1, \varphi_1) - \Phi(\kappa_2, \mathbf{w}_2, \varphi_2)]\!]_T \leq C_R(R+L)(T^{1/p'} + T^{1/p})[\![(\kappa_1, \mathbf{w}_1, \varphi_1) - (\kappa_2, \mathbf{w}_2, \varphi_2)]\!]_T$$

with some constant C_R depending on R. Choosing $T > 0$ so small that $C_R(R+L)(T^{1/p'} + T^{1/p}) \leq 1/2$, we see that Φ is a contraction map on $\mathcal{I}_{L,T}$, and therefore by the Banach fixed point theorem we have a unique fixed point $(\theta, \mathbf{v}, \pi)$ of the map Φ, which solves Eqs. (1.8) uniquely. This completes the proof of Theorem 1.2.

Acknowledgments

The author would like to thank The Directorate General of Higher Education (DGHE/DIKTI), Ministry of Research, Technology and Higher Education of Indonesia and Prof. Yoshihiro Shibata not only for insightful suggestion for this article but also for the generous support during the study.

References

[1] H. Amann, Linear and Quasilinear Parabolic Problems, Vol. I, Birkhäuser, Basel, 1995.
[2] J. Bourgain, Vector-valued singular integrals and the H^1-BMO duality, in: D. Borkholder (Ed.), Probability Theory and Harmonic Analysis, Marcel Dekker, New York, 1997, pp. 1–19.
[3] I.V. Denisova, V.A. Solonnikov, Classical solvability of a model problem in a half-space, related to the motion of an isolated mass of a compressible fluid, J. Math. Sci. 115 (2003) 2753–2765.
[4] I.V. Denisova, V.A. Solonnikov, Classical solvability of a problem on the motion of an isolated mass of a compressible liquid, St. Petersburg Math. J. 14 (2003) 1–22.
[5] R. Denk, M. Hieber, J. Prüß, R-Boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (788) (2003).
[6] Y. Enomoto, Y. Shibata, On the R-sectoriality and its applications to some mathematical study of the viscous compressible fluids, Funkcial. Ekvac. 56 (2013) 441–505.
[7] Y. Enomoto, L. von Below, Y. Shibata, On some free boundary problem for a compressible barotropic viscous fluid flow, Ann. Univ. Ferrara 60 (2014) 55–89.
[8] D. Götz, Y. Shibata, On the $\mathcal{R}$-boundedness of the solution operators in the study of the compressible viscous fluid flow with free boundary conditions, Asymptot. Anal. 90 (2014) 207–236.
[9] H. Liu, H. Yuan, J. Qiao, F. Li, Global existence of strong solutions of Navier–Stokes equations with non-Newtonian potential for one-dimensional isentropic compressible fluids, Z. Angew. Math. Phys. 63 (2012) 865–878.
[10] I. Sh. Moglievskii, Estimates of solutions of a general initial–boundary value problem for the linear nonstationary system of Navier–Stokes equations in a half-space, Zap. Nauchn. Sem. LOMI 84 (1979) 147–173 (in Russian).
[11] I. Sh. Moglievskii, Solvability of a general boundary value prolem for a lineaarized nonstationary system of Navier–Stokes equations, Zap. Nauchn. Sem. LOMI 110 (1981) 105–119 (in Russian).
[12] P.B. Mucha, W. Zajaczkowski, On the existence for the Caucy–Neumann problem for the Stokes system in the L_p-framework, Studia Math. 143 (1) (2000) 75–101.

[13] M. Murata, On a maximal L_p–L_q approach to the compressible viscous fluid flow with slip boundary condition, Nonlinear Anal. Theory Methods Appl. 106 (2014) 86–109.
[14] M. Nesensohn, Generalized viscoelastic fluids with a free boundary without surface tension, SIAM J. Math. Anal. 46 (1) (2014) 428–458.
[15] J. Prüss, G. Simonett, On the two-phase Navier–Stokes equations with surface tension, Interfaces Free Bound. 12 (2010) 311–345.
[16] J. Prüss, G. Simonett, Analytic solutions for the two-phase Navier–Stokes equations with surface tension and gravity, in: Parabolic Problems, Birkhäuser, Basel, 2011, pp. 507–540.
[17] P. Secchi, On the motion of gaseous stars in the presence of radiation, Comm. Partial Differential Equations 1 (1990) 185–204.
[18] P. Secchi, On the uniqueness of motion of viscous gaseous stars, Math. Methods Appl. Sci. 13 (1990) 391–404.
[19] P. Secchi, A. Valli, A free boundary problem for compressible viscous fluid, J. Reine Angew. Math. 341 (1983) 1–31.
[20] X. Shi, T. Wang, Z. Zhang, Asymptotic stability for one-dimensional motion of non-Newtonian compressible fluids, Acta Math. Appl. Sin. 30 (1) (2014) 99–110.
[21] Y. Shibata, On the $\mathcal{R}$-Boundedness of solution operators for the Stokes equations with free boundary condition, Differential Integral Equations 27 (2014) 313–368.
[22] Y. Shibata, On the $\mathcal{R}$-boundedness of solution operators for the weak Dirichlet–Neumann problem, in: T. Hishida (Ed.), Mathematical Analysis of Incompressible Flow February 4–6, 2013, in: RIMS Kôkyûroku, vol. 1875, Kyoto University, 2014, pp. 1–18.
[23] Y. Shibata, On some free boundary problem for the Navier–Stokes equations in the maximal L_p–L_q regularity class, J. Differential Equations 258 (2015) 4127–4155.
[24] Y. Shibata, S. Shimizu, A decay property of the Fourier Transform and its application to the Stokes problem, J. Math. Fluid Mech. 3 (3) (2001) 213–230.
[25] Y. Shibata, S. Shimizu, On the L_p–L_q Maximal regularity of the Neumann problem for the Stokes equations in a bounded domain, J. Reine Angew. Math. 615 (2008) 157–209.
[26] V.A. Solonnikov, On boundary problems for linear parabolic system of differential equations of general type, Tr. Mat. Inst. Steklova 83 (1965) 3–163 (in Russian); English transl: Proc. Steklov Inst. Math., 83 (1967), 1–184.
[27] V.A. Solonnikov, Estimates of solutions of an initial–boundary value problem for the linear non stationary Navier–Stokes system, Zap. Naunchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 59 (1976) 178–254 (in Russian).
[28] V.A. Solonnikov, Solvability of the initial–boundary-value problem for the equations of motion of a viscous compressible fluid, J. Math. Sci. 14 (1980) 1120–1133.
[29] V.A. Solonnikov, L_p-estimates for a linear problem arising in the study of the motion of an isolated liqued mass, J. Math. Sci. 189 (2013) 699–732. transl. from Problemy Matematicheskogo Analyiza 69 (82013), 137–166.
[30] V.A. Solonnikov, A. Tani, Free boundary problem for a viscous compressible flow with the surface tension, in: Ih.M. Rassias (Ed.), Constantin Carathéodory: An International Tribute, Vols. 1, 2, World Sci. Publishing, Teaneck, NJ, 1991, pp. 1270–1303.
[31] V.A. Solonnikov, A. Tani, Evolution free boundary problem for equations of motion of viscous compressible barotropic liquid, in: The Navier–Stokes equations II—Theory and Numerical Methods (Oberwalfach, 1991), in: Lecture Notes in Math., vol. 1530, Springer, Berlin, 1992, pp. 30–55.
[32] G. Ströhmer, About the resolvent of an operator from fluid dynamic, Math. Z. 194 (1987) 183–191.
[33] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann. 319 (2001) 735–758.
[34] W.M. Zajaczkowski, On nonstationary motion of a compressible barotropic viscous fluid bounded by a free surface Dissertationes Mathematicae, Polska Akademia Nauk, Inst. Mat. Warszawa, 1993.
[35] W.M. Zajaczkowski, On nonstationary motion of a compressible baratropic viscous capillary fluid bounded by a free surface, SIAM J. Math. Anal. 25 (1994) 1–84.