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ON THE $\mathcal{R}$-BOUNDEDNESS OF SOLUTION OPERATOR FAMILIES FOR TWO-PHASE STOKES RESOLVENT EQUATIONS

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Abstract. The aim of this paper is to show the existence of $\mathcal{R}$-bounded solution operator families for two-phase Stokes resolvent equations in $\dot\Omega = \Omega_+ \cup \Omega_-$, where $\Omega_\pm$ are uniform $W_r^{2-1/r}$ domains of N-dimensional Euclidean space $\mathbf{R}^N$ ($N \geq 2$, $N < r < \infty$). More precisely, given a uniform $W_r^{2-1/r}$ domain Ω with two boundaries $\Gamma_\pm$ satisfying $\Gamma_+ \cap \Gamma_- = \emptyset$, we suppose that some hypersurface Γ divides Ω into two sub-domains, that is, there exist domains $\Omega_\pm \subset \Omega$ such that $\Omega_+ \cap \Omega_- = \emptyset$ and $\Omega \setminus \Gamma = \Omega_+ \cup \Omega_-$, where $\Gamma \cap \Gamma_+ = \emptyset$, $\Gamma \cap \Gamma_- = \emptyset$, and the boundaries of $\Omega_\pm$ consist of two parts Γ and $\Gamma_\pm$, respectively. The domains $\Omega_\pm$ are filled with viscous, incompressible, and immiscible fluids with density $\rho_\pm$ and viscosity $\mu_\pm$, respectively. Here, $\rho_\pm$ are positive constants, while $\mu_\pm = \mu_\pm(x)$ are functions of $x \in \mathbf{R}^N$. On the boundaries Γ, Γ_+, and Γ_-, we consider an interface condition, a free boundary condition, and the Dirichlet boundary condition, respectively. We also show, by using the $\mathcal{R}$-bounded solution operator families, some maximal L_p-L_q regularity as well as generation of analytic semigroup for a time-dependent problem associated with the two-phase Stokes resolvent equations. This kind of problems arises in the mathematical study of the motion of two viscous, incompressible, and immiscible fluids with free surfaces. The essential assumption of this paper is the unique solvability of a weak elliptic transmission problem for $\mathbf{f} \in L_q(\Omega)^N$, that is, it is assumed that the unique existence of solutions $\theta \in \mathcal{W}_q^1(\Omega)$ to the variational problem: $(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_\Omega$ for any $\varphi \in \mathcal{W}_{q'}^1(\Omega)$ with $1 < q < \infty$ and $q' = q/(q-1)$, where ρ is defined by $\rho = \rho_+$ ($x \in \Omega_+$),

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$\rho = \rho_-$ ($x \in \Omega_-$) and $\mathcal{W}^1_q(\Omega)$ is a suitable Banach space endowed with norm $\|\cdot\|_{\mathcal{W}^1_q(\Omega)} := \|\nabla\cdot\|_{L_q(\Omega)}$. Our assumption covers e.g. the following domains as Ω: $\mathbf{R}^N$, $\mathbf{R}^N_\pm$, perturbed $\mathbf{R}^N_\pm$, layers, perturbed layers, and bounded domains, where $\mathbf{R}^N_+$ and $\mathbf{R}^N_-$ are the open upper and lower half spaces, respectively.

1. Introduction

1.1. **Problem.** Let Ω be a domain of $\mathbf{R}^N$, $N \geq 2$, with two boundaries $\Gamma_\pm$ satisfying $\Gamma_+ \cap \Gamma_- = \emptyset$. Suppose that some hypersurface Γ divides Ω into two sub-domains, that is, there exist domains $\Omega_\pm \subset \Omega$ such that $\Omega_+ \cap \Omega_- = \emptyset$ and $\Omega \setminus \Gamma = \Omega_+ \cup \Omega_-$, where $\Gamma \cap \Gamma_+ = \emptyset$, $\Gamma \cap \Gamma_- = \emptyset$, and the boundaries of $\Omega_\pm$ consist of two parts Γ and $\Gamma_\pm$, respectively. Set $\dot\Omega = \Omega_+ \cup \Omega_-$ and $\Sigma_{\varepsilon,\lambda_0} = \{\, \lambda \in \mathbf{C} : |\arg\lambda| \leq \pi - \varepsilon,\ |\lambda| \geq \lambda_0 \,\}$ for $0 < \varepsilon < \pi/2$ and $\lambda_0 > 0$. In this paper, we show the existence of $\mathcal{R}$-bounded solution operator families for the following two-phase Stokes resolvent equations with resolvent parameter λ varying in $\Sigma_{\varepsilon,\lambda_0}$:

$$\begin{cases} \lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},\theta) = \mathbf{f}, & \operatorname{div}\mathbf{u} = g \quad \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u},\theta)\mathbf{n}]\!] = [\![\mathbf{h}]\!], & [\![\mathbf{u}]\!] = 0 \quad \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u},\theta)\mathbf{n}_+ = \mathbf{k} & \quad \text{on } \Gamma_+, \\ \mathbf{u} = 0 & \quad \text{on } \Gamma_-. \end{cases} \tag{1.1}$$

Here, the unknowns $\mathbf{u} = (u_1(x), \ldots, u_N(x))^T$[1] and $\theta = \theta(x)$ are an N-vector function and a scalar function, respectively, while the right members $\mathbf{f} = (f_1(x), \ldots, f_N(x))^T$, $g = g(x)$, $\mathbf{h} = (h_1(x), \ldots, h_N(x))^T$, and $\mathbf{k} = (k_1(x), \ldots, k_N(x))^T$ are given functions. Let $\rho_\pm$ be positive constants and $\mu_\pm = \mu_\pm(x)$ scalar functions defined on $\mathbf{R}^N$, and let χ_D be the indicator function of $D \subset \mathbf{R}^N$. Then $\rho = \rho_+\chi_{\Omega_+} + \rho_-\chi_{\Omega_-}$, $\mu = \mu_+\chi_{\Omega_+} + \mu_-\chi_{\Omega_-}$, and $\mathbf{T}(\mathbf{u},\theta) = \mu\mathbf{D}(\mathbf{u}) - \theta\mathbf{I}$, where $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{D}(\mathbf{u})$ is the doubled deformation tensor, that is, the (i,j)-entry $D_{ij}(\mathbf{u})$ of $\mathbf{D}(\mathbf{u})$ is given by $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$ for $i,j = 1,\ldots,N$ and $\partial_i = \partial/\partial x_i$. In addition, $\mathbf{n}$ denotes a unit normal vector on Γ , pointing from Ω_+ to Ω_-, while $\mathbf{n}_+$ the unit outward normal vector on Γ_+. For functions f defined on $\dot\Omega$, $[\![f]\!]$ denotes a jump of f across the interface Γ as follows:

$$[\![f]\!] = [\![f]\!](x) = \lim_{y\to x,\, y\in\Omega_+} f(y) - \lim_{y\to x,\, y\in\Omega_-} f(y) \quad (x \in \Gamma).$$

[1] $\mathbf{M}^T$ denotes the transposed $\mathbf{M}$.

Here, and subsequently, we use the following symbols for differentiations: Let $f = f(x)$, $\mathbf{g} = (g_1(x), \dots, g_N(x))^T$, and $\mathbf{M} = (M_{ij}(x))$ be a scalar, an N-vector, and an $N \times N$-matrix function defined on a domain of $\mathbf{R}^N$, respectively, and then

$$\nabla f = (\partial_1 f, \dots, \partial_N f)^T, \quad \Delta f = \sum_{j=1}^N \partial_j^2 f, \quad \Delta \mathbf{g} = (\Delta g_1, \dots, \Delta g_N)^T,$$

$$\operatorname{div} \mathbf{g} = \sum_{j=1}^N \partial_j g_j, \quad \nabla^2 \mathbf{g} = \{\, \partial_i \partial_j g_k : i, j, k = 1, \dots, N \,\},$$

$$\nabla \mathbf{g} = \begin{pmatrix} \partial_1 g_1 & \dots & \partial_N g_1 \\ \vdots & \ddots & \vdots \\ \partial_1 g_N & \dots & \partial_N g_N \end{pmatrix}, \quad \operatorname{Div} \mathbf{M} = \Big(\sum_{j=1}^N \partial_j M_{1j}, \dots, \sum_{j=1}^N \partial_j M_{Nj} \Big)^T.$$

Problem (1.1) arises from a linearized system of some two-phase problem of the Navier-Stokes equations for viscous, incompressible, and immiscible fluids without taking surface tension into account. There are a lot of studies of two-phase problems for the Navier-Stokes equations. To see the history of study briefly, we restrict ourselves to the case where the two fluids are both viscous, incompressible, and immiscible in the following. Such a situation was treated in several function spaces as follows:

L_2-in-time and L_2-in-space setting. Denisova [2, 4] treated the motion of a drop Ω_{+t}, which is the region occupied by the drop at time $t > 0$, in another liquid $\Omega_{-t} = \mathbf{R}^3 \setminus \overline{\Omega_{+t}}$. More precisely, [2] showed some estimates of solutions for linearized problems and [4] an unique existence theorem local in time for the two-phase problem describing the aforementioned situation with or without surface tension. In addition, Denisova [7] proved the unique existence of global-in-time solutions for small initial data and its exponential stability in the case where Ω_{-t} is bounded and surface tension does not work. Concerning non-homogeneous incompressible fluids, Tanaka [30] showed the unique existence of global-in-time solutions for small initial data when Ω_{-t} is bounded, but surface tension is taken into account.

Hölder function spaces. A series of papers Denisova-Solonnikov [9, 10] and Denisova [3] treated the same motion as in [2, 4] mentioned above. In particular, [9] and [3] established estimates of solutions for some linearized problems, and [10] proved an unique existence theorem local in time for the two-phase problem with surface tension. On the other hand, the unique existence of global-in-time solutions for small initial data was proved by

Denisova [6] without surface tension and by Denisova-Solonnikov [11] with surface tension in the case where Ω_{-t} is bounded. Furthermore, there are other topics Denisova [5] and Denisova-Nečasová [8], which consider thermo-capillary convection and Oberbeck-Boussinesq approximation, respectively.

L_p**-in-time and** L_p**-in-space setting.** Prüss and Simonett [20, 21, 22] treated a situation that two fluids occupy $\Omega_{\pm t} = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, \pm(x_N - h(x', t)) > 0\}$, respectively, where $h(x', t)$ is an unknown scalar function describing the interface $\Gamma_t = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, x_N = h(x', t)\}$ of the fluids. [21] and [22] proved the local solvability of the two-phase problem with surface tension and with both surface tension and gravity, respectively, for small initial data. On the other hand, [20] pointed out that the Rayleigh-Taylor instability occurs if gravity works and if the fluid occupying Ω_{+t} is heavier than the other one . Furthermore, Hieber and Saito [15] extended the results of the Newtonian case of [21, 22] to a generalized Newtonian one. Köhne, Prüss, and Wilke [16] showed the local solvability and the global solvability in the case where $\Omega_{\pm t}$ are bounded and surface tension is taken into account.

L_p**-in-time and** L_q**-in-space setting.** Shibata-Shimizu [28] showed a maximal L_p-L_q regularity theorem for a linearized system of the two-phase problem considered in [20, 22] mentioned above.

This paper is a continuation of Shibata-Shimizu [28]. Our aim is in the present paper to prove the existence of $\mathcal{R}$-bounded solution operator families of (1.1) for $\dot{\Omega} = \Omega_+ \cup \Omega_-$ with uniform $W_r^{2-1/r}$ domains $\Omega_\pm$ ($N < r < \infty$), which is introduced in Definition 1.1 below. In addition, the $\mathcal{R}$-bounded solution operator families enable us to show generation of analytic semigroup and some maximal L_p-L_q regularity theorem for a time-dependent problem associated with (1.1), which are provided in Subsection 2.4 and Subsection 2.5, respectively. We want to emphasize that the maximal L_p-L_q regularity theorem extends [28] to uniform $W_r^{2-1/r}$ domains and to variable viscosities.

The strategy of this paper follows Shibata [26]. We extend his method for one-phase problem to one for two-phase problem. For example, a two-phase version of the weak Dirichlet-Neumann problem (it is called a weak elliptic transmission problem in the present paper) is introduced in Definition 1.4 below, which plays an important role in this paper, and especially in derivation of reduced Stokes resolvent equations (cf. Subsection 2.1 below) and in Lemma 5.6 below. One of the main advantage of the reduced equations is that we can eliminate the divergence equation: $\operatorname{div} \mathbf{u} = g$ in $\dot{\Omega}$, which is difficult to treat in localized problems, from the problem (1.1). On the other hand, Lemma 5.6 enable us to control localized pressure term. There, however, is a remark on Shibata's paper [26]. It seems to be difficult to obtain [26, Theorem 3.1] from [26, Theorem 3.4] and to obtain [26, Theorem 3.8] from [26, Theorem 3.10], because the $\mathcal{R}$-boundedness of $\lambda\mathfrak{g}_D(\lambda)$, $\lambda\mathfrak{g}_N(\lambda)$ was not proved in his paper (cf. [26, Proof of Theorem 3.1, Proof of Theorem 3.4]). We essentially need the $\mathcal{R}$-boundedness of such operators since the right members $\widetilde{f}$ for (3,7), (3.20) of [26] contain $\lambda V_F(g)$, $\lambda V_D(g)$, respectively. Natural spaces for ranges of the operators $\lambda\mathfrak{g}_D(\lambda)$, $\lambda\mathfrak{g}_N(\lambda)$ are given by negative Sobolev spaces, which is main difficulty to modify his proof. To overcome this difficulty, we introduce in this paper Proposition 3.5, which allows us to avoid such negative spaces. Following the strategy of Proposition 3.5, we can also complete his results.

1.2. **Notation and main results.** We first state notation used throughout this paper.

Let $\mathbf{N}$ be the set of all natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{N}_0^N$, we set $D^\alpha f = \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N} f$. Let G be an open set of $\mathbf{R}^N$. Then $L_q(G)$ and $W_q^m(G)$ with $m \in \mathbf{N}$ denote the usual $\mathbf{K}$-valued Lebesgue space and Sobolev space on G for $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, while $\|\cdot\|_{L_q(G)}$ and $\|\cdot\|_{W_q^m(G)}$ their norms, respectively. Here, we set $W_q^0(G) = L_q(G)$. In addition, $W_q^s(G)$ with $s \in (0, \infty) \setminus \mathbf{N}$ denotes the $\mathbf{K}$-valued Sobolev-Slobodezki space endowed with norm $\|\cdot\|_{W_q^s(G)}$, and also $C_0^\infty(G)$ the function space of all C^∞ functions $f : G \to \mathbf{K}$ such that $\operatorname{supp} f$ is compact and $\operatorname{supp} f \subset G$.

For two Banach spaces X and Y, $\mathcal{L}(X, Y)$ is the set of all bounded linear operators from X to Y, and $\mathcal{L}(X)$ the abbreviation of $\mathcal{L}(X, X)$. Let U be a domain of $\mathbf{C}$, and then $\operatorname{Hol}(U, \mathcal{L}(X, Y))$ stands for the set of all $\mathcal{L}(X, Y)$-valued holomorphic functions defined on U.

For $d \in \mathbf{N}$ with $d \geq 2$, X^d denotes the d-product space of X. Let $\|\cdot\|_X$ be a norm of X, while $\|\cdot\|_X$ also denotes the norm of the product space X^d for short, that is, $\|\mathbf{f}\|_X = \sum_{j=1}^d \|f_j\|_X$ for $\mathbf{f} = (f_1, \dots, f_d) \in X^d$.

Let $\mathbf{a} = (a_1, \dots, a_N)^T$ and $\mathbf{b} = (b_1, \dots, b_N)^T$, and then we write $\mathbf{a} \cdot \mathbf{b} = <\mathbf{a}, \mathbf{b}> = \sum_{j=1}^N a_j b_j$ and $\mathbf{a} \otimes \mathbf{b} = (a_i b_j)$ that is an $N \times N$ matrix with the (i, j)-entry $a_i b_j$. On the other hand, for any vector functions $\mathbf{u}$, $\mathbf{v}$ on G, we set $(\mathbf{u}, \mathbf{v})_G = \int_G \mathbf{u} \cdot \mathbf{v}\, dx$ and $(\mathbf{u}, \mathbf{v})_{\partial G} = \int_{\partial G} \mathbf{u} \cdot \mathbf{v}\, d\sigma$, where ∂G is the boundary of G and $d\sigma$ the surface element of ∂G.

Given $1 < q < \infty$, we set $q' = q/(q-1)$. Let $L_{q,\mathrm{loc}}(\overline{G})$ be the vector space of all measurable functions f: $G \to \mathbf{K}$ such that $f \in L_q(G \cap B)$ for any ball B of $\mathbf{R}^N$. We define a homogeneous space $\widehat{W}_q^1(G)$ by $\widehat{W}_q^1(G) = \{f \in L_{q,\mathrm{loc}}(\overline{G}) : \nabla f \in L_q(G)^N\}$ with norm $\|\cdot\|_{\widehat{W}_q^1(G)} := \|\nabla\cdot\|_{L_q(G)}$, where we have to identify two elements differing by a constant. In addition, let $\widehat{W}_{q,0}^1(G)$ and $W_{q,0}^1(G)$ be Banach spaces defined by $X_{q,0}^1(G) = \{f \in X_q^1(G) : f = 0 \text{ on } \partial G\}$ $(X \in \{\widehat{W}, W\})$ with norms $\|\cdot\|_{\widehat{W}_{q,0}^1(G)} := \|\nabla\cdot\|_{L_q(G)}$ and $\|\cdot\|_{W_{q,0}^1(G)} := \|\cdot\|_{W_q^1(G)}$, respectively.

Throughout this paper, the letter C denotes generic constants and $C_{a,b,c,\ldots}$ means that the constant depends on the quantities a, b, c, $\ldots$. The values of constants C and $C_{a,b,c,\ldots}$ may change from line to line.

Secondly, we introduce some definitions.

Definition 1.1 (Uniform $W_r^{2-1/r}$ domains)**.** *Let $1 < r < \infty$ and D be a domain of $\mathbf{R}^N$ with boundary ∂D. We say that D is a uniform $W_r^{2-1/r}$ domain, if there exist positive constants α, β, and K such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \partial D$ there are a coordinate number j and a $W_r^{2-1/r}$ function $h(x')$ $(x' = (x_1, \ldots, \widehat{x}_j, \ldots, x_N))$ defined on $B'_\alpha(x'_0)$, with $x'_0 = (x_{01}, \ldots, \widehat{x}_{0j}, \ldots, x_{0N})$ and $\|h\|_{W_r^{2-1/r}(B'_\alpha(x'_0))} \leq K$, such that*

$$D \cap B_\beta(x_0) = \{x \in \mathbf{R}^N : x_j > h(x'),\, x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0),$$
$$\partial D \cap B_\beta(x_0) = \{x \in \mathbf{R}^N : x_j = h(x'),\, x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0).$$

Here, $(x_1, \ldots, \widehat{x}_j, \ldots, x_N) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$, $B'_\alpha(x'_0) = \{x' \in \mathbf{R}^{N-1} : |x' - x'_0| < \alpha\}$, and $B_\beta(x_0) = \{x \in \mathbf{R}^N : |x - x_0| < \beta\}$.

Definition 1.2 ($\mathcal{R}$-boundedness)**.** *Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that the following assertion holds: For each natural number n, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and for all sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, there holds the inequality:*

$$\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(u) T_j f_j\Big\|_Y^p\, du\Big)^{1/p} \leq C\Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(u) f_j\Big\|_X^p\, du\Big)^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}$.

Remark 1.3. The constant C in Definition 1.2 depends on p. On the other hand, it is well-known that $\mathcal{T}$ is $\mathcal{R}$-bounded for any $p \in [1, \infty)$, provided that $\mathcal{T}$ is $\mathcal{R}$-bounded for some $p \in [1, \infty)$. This fact follows from Kahane's inequality (cf. [18, Theorem 2.4]).

Furthermore, we introduce a weak elliptic transmission problem. In the present paper, $\Gamma_+ = \emptyset$ or $\Gamma_- = \emptyset$ are admissible, but note that $\Gamma \neq \emptyset$. Let $W^1_{q,\Gamma_+}(\Omega)$ and $\widehat{W}^1_{q,\Gamma_+}(\Omega)$ be Banach spaces defined by

$$X^1_{q,\Gamma_+}(\Omega) = \begin{cases} \{f \in X^1_q(\Omega) : f = 0 \text{ on } \Gamma_+\} & \text{if } \Gamma_+ \neq \emptyset, \\ X^1_q(\Omega) & \text{if } \Gamma_+ = \emptyset, \end{cases}$$

for $X \in \{W, \widehat{W}\}$, and their norms are given by $\|\cdot\|_{W^1_{q,\Gamma_+}(\Omega)} = \|\cdot\|_{W^1_q(\Omega)}$ and $\|\cdot\|_{\widehat{W}^1_{q,\Gamma_+}(\Omega)} = \|\nabla\cdot\|_{L_q(\Omega)}$, respectively. The unique solvability of the weak elliptic transmission problem is defined as follows:

Definition 1.4. *Let $1 < q < \infty$ and $q' = q/(q-1)$. Let $\mathcal{W}^1_q(\Omega)$ be a closed subspace of $\widehat{W}^1_{q,\Gamma_+}(\Omega)$, and suppose that $W^1_{q,\Gamma_+}(\Omega)$ is dense in $\mathcal{W}^1_q(\Omega)$. Set $\rho = \rho_+\chi_{\Omega_+} + \rho_-\chi_{\Omega_-}$ for positive constants $\rho_\pm$. Then, we say that the weak elliptic transmission problem is uniquely solvable on $\mathcal{W}^1_q(\Omega)$ for $\rho_\pm$ if the following assertion holds: For any $\mathbf{f} \in L_q(\Omega)^N$, there is a unique $\theta \in \mathcal{W}^1_q(\Omega)$ satisfying the variational equation:*

$$(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot{\Omega}} = (\mathbf{f}, \nabla\varphi)_\Omega \quad \textit{for all } \varphi \in \mathcal{W}^1_{q'}(\Omega),$$

which possesses the estimate: $\|\nabla\theta\|_{L_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$ with a positive constant C independent of θ, φ, and $\mathbf{f}$.

Remark 1.5. (1) Let $1 < q < \infty$, $q' = q/(q-1)$, and let the weak elliptic transmission problem be uniquely solvable on $\mathcal{W}^1_q(\Omega)$ for $\rho_+ = \rho_- = 1$. We define $J_q(\Omega)$ and $G_q(\Omega)$ by

$$J_q(\Omega) = \{\mathbf{f} \in L_q(\Omega)^N : (\mathbf{f}, \nabla\varphi)_\Omega = 0 \quad \text{for all } \varphi \in \mathcal{W}^1_{q'}(\Omega)\},$$
$$G_q(\Omega) = \{\mathbf{f} \in L_q(\Omega)^N : \mathbf{f} = \nabla\theta \quad \text{for some } \theta \in \mathcal{W}^1_q(\Omega)\}.$$

Then, by the standard proof, the so-called *Helmholtz decomposition*: $L_q(\Omega)^N = J_q(\Omega) \oplus G_q(\Omega)$ holds.

(2) In applications, we choose $\mathcal{W}^1_q(\Omega)$ in such a way that the weak elliptic transmission problem is uniquely solvable for $\rho_\pm$. Typical examples are as follows: $\mathcal{W}^1_q(\mathbf{R}^N) = \widehat{W}^1_q(\mathbf{R}^N)$; $\mathcal{W}^1_q(\mathbf{R}^N_+) = \widehat{W}^1_q(\mathbf{R}^N_+)$ with $\Gamma_+ = \emptyset$ and $\Gamma_- = \mathbf{R}^N_0 = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, x_N = 0\}$, and $\mathcal{W}^1_q(\Omega)$ is analogously defined by $\widehat{W}^1_q(\Omega)$ when Ω is a perturbed $\mathbf{R}^N_+$; $\mathcal{W}^1_q(\mathbf{R}^N_-) = \widehat{W}^1_{q,\Gamma_+}(\mathbf{R}^N_-)$ with $\Gamma_+ = \mathbf{R}^N_0$ and $\Gamma_- = \emptyset$, and $\mathcal{W}^1_q(\Omega)$ is analogously defined by

$\widehat{W}^1_{q,\Gamma_+}(\Omega)$ when Ω is a perturbed $\mathbf{R}^N_-$; $\mathcal{W}^1_q(\Omega) = \widehat{W}^1_{q,\Gamma_+}(\Omega)$ when Ω is a bounded domain, a layer, or a perturbed layer. We refer e.g. to [16, Appendix A.1] for the treatment of weak elliptic transmission problems.

(3) We set $W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega) = \{\theta_1 + \theta_2 : \theta_1 \in W^1_q(\dot\Omega),\, \theta_2 \in \mathcal{W}^1_q(\Omega)\}$. Suppose that the weak elliptic transmission problem is uniquely solvable on $\mathcal{W}^1_q(\Omega)$ for $\rho_\pm$. Then, for any $\alpha \in L_q(\dot\Omega)^N$, $\beta \in W^{1-1/q}_q(\Gamma)$, and $\gamma \in W^{1-1/q}_q(\Gamma_+)$, there exists a unique $\theta \in W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega)$ satisfying the weak problem: for all $\varphi \in \mathcal{W}^1_q(\Omega)$,

$$(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot\Omega} = (\alpha, \nabla\varphi)_{\dot\Omega}, \quad [\![\theta]\!] = \beta \quad \text{on } \Gamma, \quad \theta = \gamma \quad \text{on } \Gamma_+, \tag{1.2}$$

which possesses the estimate:

$$\|\nabla\theta\|_{L_q(\dot\Omega)} \leq C\left(\|\alpha\|_{L_q(\dot\Omega)} + \|\beta\|_{W^{1-1/q}_q(\Gamma)} + \|\gamma\|_{W^{1-1/q}_q(\Gamma_+)}\right),$$

with some positive constant C independent of α, β, γ, θ, and φ. Thus, it is possible to define a linear operator

$$\mathcal{K} : L_q(\dot\Omega)^N \times W^{1-1/q}_q(\Gamma) \times W^{1-1/q}_q(\Gamma_+) \to W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega),$$

by $\mathcal{K}(\alpha,\beta,\gamma) = \theta$ satisfying (1.2). If $\Gamma_+ = \emptyset$, then we denote $\mathcal{K}(\alpha,\beta,\gamma)$ by $\mathcal{K}(\alpha,\beta,\emptyset)$ when $\Gamma_- \neq \emptyset$ and by $\mathcal{K}(\alpha,\beta)$ when $\Gamma_- = \emptyset$.

We now state our main results. To this end, we introduce a data space for the divergence equation: $\operatorname{div}\mathbf{u} = g$ in $\dot\Omega$ with boundary conditions: $[\![\mathbf{u}]\!]\cdot\mathbf{n} = 0$ on Γ and $\mathbf{u}\cdot\mathbf{n}_- = 0$ on Γ_-, where $\mathbf{n}_-$ is the unit outward normal vector on Γ_-. Let $\mathcal{W}^{-1}_q(\Omega)$ be the dual space of $\mathcal{W}^1_{q'}(\Omega)$ for $1 < q < \infty$ and $q' = q/(q-1)$, and let $\|\cdot\|_{\mathcal{W}^{-1}_q(\Omega)}$ and $< \cdot,\cdot >_\Omega$ be its norm and the duality pairing between $\mathcal{W}^{-1}_q(\Omega)$ and $\mathcal{W}^1_{q'}(\Omega)$, respectively. Then, we set $L_q(\Omega) \cap \mathcal{W}^{-1}_q(\Omega) = \big\{g \in L_q(\Omega) : \exists M > 0 \text{ s.t. } |(g,\varphi)_\Omega| \leq M\|\nabla\varphi\|_{L_{q'}(\Omega)}, \text{ for any } \varphi \in W^1_{q',\Gamma_+}(\Omega)\big\}$. Let $g \in L_q(\Omega) \cap \mathcal{W}^{-1}_q(\Omega)$, and thus g can be extended uniquely to an element of $\mathcal{W}^{-1}_q(\Omega)$. Such an extended g is again denoted by g for short. We can see g as a functional on $\{\nabla\theta : \theta \in \mathcal{W}^1_{q'}(\Omega)\} \subset L_{q'}(\Omega)^N$, which, combined with Hahn-Banach's theorem, furnishes that there is a $\mathbf{G} \in L_q(\Omega)^N$ such that $\|g\|_{\mathcal{W}^{-1}_q(\Omega)} = \|\mathbf{G}\|_{L_q(\Omega)}$ and $< g, \varphi >_\Omega = -(\mathbf{G}, \nabla\varphi)_\Omega$ for all $\varphi \in \mathcal{W}^1_{q'}(\Omega)$. In what follows, $\mathbf{G}$ is restricted to the functional on $\{\nabla\theta : \theta \in \mathcal{W}^1_{q'}(\Omega)\}$. Let $\widetilde{L}_q(\Omega) = L_q(\Omega)^N/J_q(\Omega)$, and let $[\mathbf{G}] = \{\mathbf{G} + \mathbf{f} : \mathbf{f} \in J_q(\Omega)\} \in \widetilde{L}_q(\Omega)$. Then $g \mapsto [\mathbf{G}]$ is well-defined, so that we denote $[\mathbf{G}]$ by $\mathcal{G}(g)$. Especially, we have, for $g \in L_q(\Omega) \cap \mathcal{W}_q^{-1}(\Omega)$ and for any representative $\mathfrak{g} \in L_q(\Omega)^N$ of $\mathcal{G}(g)$,

$$(g, \varphi)_\Omega = -(\mathfrak{g}, \nabla\varphi)_\Omega \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega). \tag{1.3}$$

We here set

$$\mathbf{W}_q^{-1}(\Omega) = L_q(\Omega) \cap \mathcal{W}_q^{-1}(\Omega).$$

Then, $W_q^1(\dot\Omega) \cap \mathbf{W}_q^{-1}(\Omega)$ is a Banach space with norm $\|\cdot\|_{W_q^1(\dot\Omega)\cap\mathbf{W}_q^{-1}(\Omega)} := \|\cdot\|_{W_q^1(\dot\Omega)} + \|\cdot\|_{\mathcal{W}_q^{-1}(\Omega)}$, and the function space is characterized as the data space for the divergence equation above. The following theorem presents the main result of this paper.

Theorem 1.6. *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $N < r < \infty$, and $\max(q, q') \le r$ with $q' = q/(q-1)$. Let $\rho_\pm$ be positive constants. Suppose that the following three conditions holds:*

(a) *$\Omega_\pm$ are uniform $W_r^{2-1/r}$ domains;*

(b) *The weak elliptic transmission problem is uniquely solvable on $\mathcal{W}_q^1(\Omega)$ and $\mathcal{W}_{q'}^1(\Omega)$ for $\rho_\pm$;*

(c) *$\mu_\pm$ are real valued uniformly continuous functions defined on $\mathbf{R}^N$ and there exist positive constants $\mu_{\pm1}$, $\mu_{\pm2}$ such that*

$$\mu_{+1} \le \mu_+(x) \le \mu_{+2}, \quad \mu_{-1} \le \mu_-(x) \le \mu_{-2} \quad \textit{for any } x \in \mathbf{R}^N.$$

In addition, $\mu_\pm \in W^1_{r,\mathrm{loc}}(\mathbf{R}^N)$ and $\|\nabla\mu_\pm\|_{L_r(B)} \le K_{r,\tau}$ with some positive constant $K_{r,\tau}$ for any ball $B \subset \mathbf{R}^N$ with radius τ.

(1) **Existence.** *Set*

$$\begin{aligned} X_q = \{(\mathbf{f}, g, \mathbf{h}, \mathbf{k}) : \mathbf{f} \in L_q(\dot\Omega)^N,\ g \in W_q^1(\dot\Omega) \cap \mathbf{W}_q^{-1}(\Omega),\\ \mathbf{h} \in W_q^1(\dot\Omega)^N,\ \mathbf{k} \in W_q^1(\Omega_+)^N\},\\ \mathcal{X}_q = \{(F_1, \dots, F_{11}) : F_1, F_2, F_4, F_7 \in L_q(\dot\Omega)^N,\ F_3 \in L_q(\dot\Omega),\\ F_5 \in W_q^1(\dot\Omega),\ F_6 \in L_q(\dot\Omega)^{N^2},\ F_8 \in W_q^1(\dot\Omega)^N,\\ F_9 \in L_q(\Omega_+)^{N^2},\ F_{10} \in L_q(\Omega_+)^N,\ F_{11} \in W_q^1(\Omega_+)^N\}. \end{aligned}$$

Then, there exists a constant $\lambda_0 \ge 1$ and operator families:

$$\begin{aligned} \mathbf{A}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q, W_q^2(\dot\Omega)^N)),\\ \mathbf{P}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q, W_q^1(\dot\Omega) + \mathcal{W}_q^1(\Omega))), \end{aligned}$$

such that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and for any $(\mathbf{f}, g, \mathbf{h}, \mathbf{k}) \in X_q$ and $\mathfrak{g} \in \mathcal{G}(g)$, $\mathbf{u} = \mathbf{A}(\lambda)F_\lambda(\mathbf{f}, g, \mathfrak{g}, \mathbf{h}, \mathbf{k})$ and $\theta = \mathbf{P}(\lambda)F_\lambda(\mathbf{f}, g, \mathfrak{g}, \mathbf{h}, \mathbf{k})$ are solutions to

(1.1)*, and furthermore,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\dot\Omega)^{\widetilde N})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (R_\lambda \mathbf{A}(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \leq \gamma_0,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\dot\Omega)^{N})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l \nabla \mathbf{P}(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \leq \gamma_0,$$

for $l = 0, 1$ *with some positive constant* γ_0*. Here, we have set*

$$\widetilde N = N^3 + N^2 + N, \quad R_\lambda \mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u}),$$

and

$$F_\lambda(\mathbf{f}, g, \mathfrak{g}, \mathbf{h}, \mathbf{k}) = (\mathbf{f}, \nabla g, \lambda^{1/2} g, \lambda\mathfrak{g}, g, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h}, \nabla\mathbf{k}, \lambda^{1/2}\mathbf{k}, \mathbf{k}).$$

(2) **Uniqueness.** *There exists a* $\lambda_0 \geq 1$ *such that if* $\mathbf{u} \in W^2_q(\dot\Omega)^N \cap J_q(\Omega)$ *and* $\theta \in W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega)$ *satisfies the homogeneous equations:*

$$\lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, \theta) = 0 \quad \text{in } \dot\Omega, \quad [\![\mathbf{T}(\mathbf{u}, \theta)\mathbf{n}]\!] = 0, \quad [\![\mathbf{u}]\!] = 0 \quad \text{on } \Gamma,$$
$$\mathbf{T}(\mathbf{u}, \theta)\mathbf{n}_+ = 0 \quad \text{on } \Gamma_+, \quad \mathbf{u} = 0 \quad \text{on } \Gamma_-$$

with $\lambda \in \Sigma_{\varepsilon,\lambda_0}$*, then* $\mathbf{u} = 0$ *in* $\dot\Omega$*.*

Remark 1.7. The symbols F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, and F_{11} are variables corresponding to $\mathbf{f}$, ∇g, $\lambda^{1/2}g$, $\lambda\mathfrak{g}$, g, $\nabla\mathbf{h}$, $\lambda^{1/2}\mathbf{h}$, $\mathbf{h}$, $\nabla\mathbf{k}$, $\lambda^{1/2}\mathbf{k}$, and $\mathbf{k}$, respectively. The norm of space $\mathcal{X}_q$ is given by

$$\|(F_1, \ldots, F_{11})\|_{\mathcal{X}_q} = \|(F_1, F_2, F_3, F_4, F_6, F_7)\|_{L_q(\dot\Omega)} + \|(F_5, F_8)\|_{W^1_q(\dot\Omega)}$$
$$+ \|(F_9, F_{10})\|_{L_q(\Omega_+)} + \|F_{11}\|_{W^1_q(\Omega_+)}.$$

This paper is organized as follows: The next section first tells us some equivalence of (1.1) and two-phase reduced Stokes resolvent equations, which are obtained by elimination of pressure term from (1.1), in Subsection 2.1 and Subsection 2.2. Secondly, we state our main result for the two-phase reduced Stokes resolvent equations in Subsection 2.3, which, combined with what pointed out in Subsection 2.2, allows us to conclude that Theorem 1.6 holds. Thirdly, we state generation of analytic semigroup and some maximal L_p-L_q regularity theorem for two-phase problems of time-dependent Stokes equations in Subsection 2.4 and Subsection 2.5, respectively, with help of Theorem 1.6 and the main result stated in Subsection 2.3. Section 3 proves our main result for the two-phase reduced Stokes resolvent equations in the case where

$$\dot\Omega = \dot{\mathbf{R}}^N = \mathbf{R}^N_+ \cup \mathbf{R}^N_-, \quad \mathbf{R}^N_\pm = \{(x', x_N) : x' \in \mathbf{R}^{N-1}, \pm x_N > 0\},$$

with constant viscosity coefficients. Section 4 proves our main result for the two-phase reduced Stokes resolvent equations with variable viscosity coefficients when $\dot\Omega$ is a perturbed $\dot{\mathbf{R}}^N$ by using results obtained in Section 3. Section 5 shows the main result stated in Subsection 2.3 by using results obtained in Section 4 together with some localization technique.

2. Generation of analytic semigroup and maximal regularity

In this section, after introducing the Stokes operator in (2.15) below, we consider the following initial-boundary value problem:

$$\begin{cases} \partial_t\mathbf{v} - \rho^{-1}\,\mathrm{Div}\,\mathbf{T}(\mathbf{v},\pi) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{v} = g & \text{in } \dot\Omega\times(0,\infty),\\ [\![\mathbf{T}(\mathbf{v},\pi)\mathbf{n}]\!] = [\![\mathbf{h}]\!] \quad [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma\times(0,\infty),\\ \mathbf{T}(\mathbf{v},\pi)\mathbf{n}_+ = \mathbf{k} & \text{on } \Gamma_+\times(0,\infty),\\ \mathbf{v} = 0 & \text{on } \Gamma_-\times(0,\infty),\\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \dot\Omega, \end{cases} \tag{2.1}$$

which is called the two-phase Stokes equations in this paper. We discuss the generation of analytic semigroup associated with (2.1) and some maximal L_p-L_q regularity property for (2.1). To consider the generation of analytic semigroup, we have to formulate (2.1) in the semigroup setting, that is, we have to eliminate the pressure term from (2.1). Throughout this section, for some $1<q<\infty$ and positive constants $\rho_\pm$, we assume that the weak elliptic transmission problem is uniquely solvable on $\mathcal{W}^1_q(\Omega)$ for $\rho_\pm$. The assumption plays an essential role to eliminate the pressure term from (2.1).

2.1. **Two-phase reduced Stokes resolvent equations.** Let $1<q<\infty$, $q'=q/(q-1)$, and $\mathbf{u}\in W^2_q(\dot\Omega)^N$. Set $K(\mathbf{u}) = \mathcal{K}(\alpha,\beta,\gamma)\in W^1_q(\dot\Omega)+\mathcal{W}^1_q(\Omega)$, defined in Remark 1.5 (3), with

$$\alpha = \rho^{-1}\,\mathrm{Div}(\mu\mathbf{D}(\mathbf{u})) - \nabla\,\mathrm{div}\,\mathbf{u}, \quad \beta = <[\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}]\!],\mathbf{n}> - [\![\mathrm{div}\,\mathbf{u}]\!], \tag{2.2}$$
$$\gamma = <\mu\mathbf{D}(\mathbf{u})\mathbf{n}_+,\mathbf{n}_+> - \mathrm{div}\,\mathbf{u}.$$

Then $\mathbf{u}\mapsto\nabla K(\mathbf{u})$ is a bounded linear operator from $W^2_q(\dot\Omega)^N$ to $L_q(\dot\Omega)^N$ with $\|\nabla K(\mathbf{u})\|_{L_q(\dot\Omega)}\le C\|\mathbf{u}\|_{W^2_q(\dot\Omega)}$ for some positive constant C independent of $\mathbf{u}$. We consider the equations as follows:

$$\begin{cases} \lambda\mathbf{u} - \rho^{-1}\,\mathrm{Div}\,\mathbf{T}(\mathbf{u}, K(\mathbf{u})) = \mathbf{f} & \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}]\!] = [\![\mathbf{h}]\!] & \text{on } \Gamma, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}_+ = \mathbf{k} & \text{on } \Gamma_+, \\ \mathbf{u} = 0 & \text{on } \Gamma_-, \end{cases} \tag{2.3}$$

which is called the two-phase reduced Stokes resolvent equations. In this subsection, we construct a solution to (2.3) on the assumption that (1.1) is solvable. To this end, we treat the following auxiliary problem:

$$(\lambda u, \varphi)_{\dot\Omega} + (\nabla u, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \tag{2.4}$$

$$[\![u]\!] = g \quad \text{on } \Gamma, \quad u = h \quad \text{on } \Gamma_+, \tag{2.5}$$

which is the weak elliptic transmission problem with resolvent parameter λ. Employing the same argument as in the proof of our main result in the present paper, we can show the following proposition.

Proposition 2.1. *Let* $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $N < r < \infty$, *and* $\max(q, q') \leq r$ *with* $q' = q/(q-1)$. *Suppose that* $\Omega_\pm$ *are uniform* $W^{2-1/r}_r$ *domains. Then there is a positive number* $\lambda_0 \geq 1$ *such that, for any* $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, $\mathbf{f} \in L_q(\dot\Omega)^N$, $g \in W^{1-1/q}_q(\Gamma)$, *and* $h \in W^{1-1/q}_q(\Gamma_+)$, (2.4)-(2.5) *admit a unique solution* $u \in W^1_q(\dot\Omega) \cap \mathbf{W}^{-1}_q(\Omega)$.

We solve (2.3) by means of solutions to (1.1). Given $\mathbf{f} \in L_q(\dot\Omega)^N$, $\mathbf{h} \in W^1_q(\dot\Omega)^N$, and $\mathbf{k} \in W^1_q(\Omega_+)^N$, we choose by Proposition 2.1 some g in such a way that g solves the weak problem:

$$(\lambda g, \varphi)_{\dot\Omega} + (\nabla g, \nabla\varphi)_{\dot\Omega} = -(\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \tag{2.6}$$

$$[\![g]\!] = < [\![\mathbf{h}]\!], \mathbf{n} > \quad \text{on } \Gamma, \quad g = < \mathbf{k}, \mathbf{n}_+ > \quad \text{on } \Gamma_+. \tag{2.7}$$

Let $\mathbf{u} \in W^2_q(\dot\Omega)^N$ and $\theta \in W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega)$ be solutions to (1.1) with $\mathbf{f}$, g, $\mathbf{h}$, and $\mathbf{k}$ as above. Then, by the definition of $K(\mathbf{u})$ and Gauss's divergence theorem together with $[\![\mathbf{u}]\!] = 0$ on Γ, $\mathbf{u} = 0$ on Γ_-, we see that

$$(\mathbf{f}, \nabla\varphi)_{\dot\Omega} = -(\lambda g, \varphi)_{\dot\Omega} - (\nabla g, \nabla\varphi)_{\dot\Omega} + (\rho^{-1}\nabla(\theta - K(\mathbf{u})), \nabla\varphi)_{\dot\Omega},$$

for any $\varphi \in W^1_{q,\Gamma_+}(\Omega)$. This combined with (2.6) and the denseness of $W^1_{q',\Gamma_+}(\Omega)$ in $\mathcal{W}^1_{q'}(\Omega)$ furnishes that

$$(\rho^{-1}\nabla(\theta - K(\mathbf{u})), \nabla\varphi)_{\dot\Omega} = 0 \quad \text{for all } \varphi \in \mathcal{W}^1_{q'}(\Omega).$$

In addition, it holds that $[\![K(\mathbf{u})-\theta]\!]=0$ on Γ and $K(\mathbf{u})-\theta=0$ on Γ_+, since g satisfies (2.7) and

$$<[\![\mathbf{h}]\!],\mathbf{n}>=[\![K(\mathbf{u})-\theta]\!]+[\![g]\!] \quad \text{on } \Gamma, \quad <\mathbf{k},\mathbf{n}_+>=K(\mathbf{u})-\theta+g \quad \text{on } \Gamma_+.$$

Thus, the unique solvability of the weak elliptic transmission problem implies $K(\mathbf{u})=\theta$, which means that the solution $\mathbf{u}\in W_q^2(\dot\Omega)^N$ of (1.1) solves (2.3) for $\mathbf{f}\in L_q(\dot\Omega)^N$, $\mathbf{h}\in W_q^1(\dot\Omega)^N$, $\mathbf{k}\in W_q^1(\Omega_+)^N$, and g of (2.6)-(2.7).

2.2. **Reduced Stokes implies Stokes.** In this subsection, we solve (1.1) on the assumption that (2.3) is solvable. Let $1<q<\infty$ and $q'=q/(q-1)$. Given $\mathbf{f}\in L_q(\dot\Omega)^N$, $\mathbf{h}\in W_q^1(\dot\Omega)^N$, and $\mathbf{k}\in W_q^1(\Omega_+)^N$, let $\kappa\in W_q^1(\dot\Omega)+\mathcal{W}_q^1(\Omega)$ be a solution to the weak problem:

$$\begin{aligned}&(\rho^{-1}\nabla\kappa,\nabla\varphi)_{\dot\Omega}=(\mathbf{f},\nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi\in\mathcal{W}_{q'}^1(\Omega),\\ &[\![\kappa]\!]=-<[\![\mathbf{h}]\!],\mathbf{n}> \quad \text{on } \Gamma, \quad \kappa=-<\mathbf{k},\mathbf{n}_+> \quad \text{on } \Gamma_+.\end{aligned}$$

Then, the system (1.1) is reduced to

$$\begin{cases}\lambda\mathbf{u}-\rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},\theta-\kappa)=\mathbf{f}-\rho^{-1}\nabla\kappa, & \operatorname{div}\mathbf{u}=g \quad \text{in } \dot\Omega,\\ [\![\mathbf{T}(\mathbf{u},\theta-\kappa)\mathbf{n}]\!]=[\![\mathbf{h}]\!]-<[\![\mathbf{h}]\!],\mathbf{n}>\mathbf{n}, & [\![\mathbf{u}]\!]=0 \quad \text{on } \Gamma,\\ \mathbf{T}(\mathbf{u},\theta-\kappa)\mathbf{n}_+=\mathbf{k}-<\mathbf{k},\mathbf{n}_+>\mathbf{n}_+ & \text{on } \Gamma_+,\\ \mathbf{u}=0 & \text{on } \Gamma_-.\end{cases}$$

It thus suffices to consider (1.1) under the condition that

$$\begin{aligned}&(\mathbf{f},\nabla\varphi)_{\dot\Omega}=0 \quad \text{for all } \varphi\in\mathcal{W}_{q'}^1(\Omega),\\ &<[\![\mathbf{h}]\!],\mathbf{n}>=0 \quad \text{on } \Gamma, \quad <\mathbf{k},\mathbf{n}_+>=0 \quad \text{on } \Gamma_+.\end{aligned} \tag{2.8}$$

For $\mathbf{G}=(G_1,G_2)\in L_q(\dot\Omega)^N\times W_q^1(\dot\Omega)$, we set $L(\mathbf{G})=L(G_1,G_2)=\mathcal{K}(G_1-\nabla G_2,-[\![G_2]\!],-G_2)$ by $\mathcal{K}$ of Remark 1.5 (3). Then $\mathbf{G}\mapsto\nabla L(\mathbf{G})$ is a bounded linear operator from $L_q(\dot\Omega)^N\times W_q^1(\dot\Omega)$ to $L_q(\dot\Omega)^N$.

Given $g\in W_q^1(\dot\Omega)\cap\mathbf{W}_q^{-1}(\Omega)$, we choose a representative $\mathfrak{g}$ of $\mathcal{G}(g)$. For these g, $\mathfrak{g}$ and for $\mathbf{f}$, $\mathbf{h}$, $\mathbf{k}$ satisfying (2.8), let $\mathbf{u}\in W_q^2(\dot\Omega)^N$ be a solution to the two-phase reduced Stokes resolvent equations as follows:

$$\begin{cases}\lambda\mathbf{u}-\rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},K(\mathbf{u}))=\mathbf{f}+\rho^{-1}\nabla L(\lambda\mathfrak{g},g) & \text{in } \dot\Omega,\\ [\![\mathbf{T}(\mathbf{u},K(\mathbf{u}))\mathbf{n}]\!]=[\![\mathbf{h}]\!]+[\![g]\!]\mathbf{n} & \text{on } \Gamma,\\ [\![\mathbf{u}]\!]=0 & \text{on } \Gamma,\\ \mathbf{T}(\mathbf{u},K(\mathbf{u}))\mathbf{n}_+=\mathbf{k}+g\mathbf{n}_+ & \text{on } \Gamma_+,\\ \mathbf{u}=0 & \text{on } \Gamma_-.\end{cases}$$

Then, by (1.3), (2.8) and by the definition of $K(\mathbf{u})$, $L(\lambda\mathfrak{g}, g)$, we have

$$0 = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} = (\lambda\mathbf{u}, \nabla\varphi)_{\dot\Omega} - (\nabla \operatorname{div}\mathbf{u}, \nabla\varphi)_{\dot\Omega} + (\lambda g, \varphi)_{\dot\Omega} + (\nabla g, \nabla\varphi)_{\dot\Omega},$$

for any $\varphi \in W^1_{q,\Gamma_+}(\Omega)$, which, combined with Gauss's divergence theorem together with $[\![\mathbf{u}]\!] = 0$ on Γ and $\mathbf{u} = 0$ on Γ_-, furnishes that $(\lambda(\operatorname{div}\mathbf{u} - g), \varphi)_{\dot\Omega} + (\nabla(\operatorname{div}\mathbf{u} - g), \nabla\varphi)_{\dot\Omega} = 0$ for all $\varphi \in W^1_{q',\Gamma_+}(\Omega)$. In addition, we see, by (2.8) and the definition of $K(\mathbf{u})$, that $[\![g]\!] =< [\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}]\!], \mathbf{n} > -[\![K(\mathbf{u})]\!] = [\![\operatorname{div}\mathbf{u}]\!]$ on Γ, $g =< \mu\mathbf{D}(\mathbf{u})\mathbf{n}_+, \mathbf{n}_+ > -K(\mathbf{u}) = \operatorname{div}\mathbf{u}$ on Γ_+, which implies that $[\![\operatorname{div}\mathbf{u} - g]\!] = 0$ on Γ, $\operatorname{div}\mathbf{u} - g = 0$ on Γ_+. Thus, by Proposition 2.1, $\operatorname{div}\mathbf{u} = g$ in $\dot\Omega$, which means that $\mathbf{u}$ and $\theta = K(\mathbf{u}) - L(\lambda\mathfrak{g}, g)$ solves (1.1).

2.3. **$\mathcal{R}$-bounded solution operator families of reduced Stokes.** According to what was pointed out in Subsection 2.1 and Subsection 2.2, we consider the two-phase reduced Stokes resolvent equations (2.3) instead of (1.1) from Section 3 through Section 5. More precisely, we prove

Theorem 2.2. *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $N < r < \infty$, and $\max(q, q') \le r$ with $q' = q/(q-1)$. Let $\rho_\pm$ be positive constants. Suppose that* (a), (b), *and* (c) *stated in Theorem* 1.6 *hold. For any open set G of $\mathbf{R}^N$, let $X_{\mathcal{R},q}(G)$ and $\mathcal{X}_{\mathcal{R},q}(G)$ be defined as*

$$X_{\mathcal{R},q}(G) = \{(\mathbf{f}, \mathbf{h}, \mathbf{k}) : \mathbf{f} \in L_q(G)^N,\ \mathbf{h} \in W^1_q(G)^N,\ \mathbf{k} \in W^1_q(G \cap \Omega_+)^N\},$$

$$\begin{aligned}\mathcal{X}_{\mathcal{R},q}(G) = \{(H_1, \ldots, H_7) : {}& H_1, H_3 \in L_q(G)^N, H_2 \in L_q(G)^{N^2}, H_4 \in W^1_q(G)^N,\\ & H_5 \in L_q(G \cap \Omega_+)^{N^2},\ H_6 \in L_q(G \cap \Omega_+)^N,\ H_7 \in W^1_q(G \cap \Omega_+)^N\}.\end{aligned}$$

Then there exist a positive number $\lambda_0 \ge 1$ and an operator family $\mathbf{B}(\lambda) \in \operatorname{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), W^2_q(\dot\Omega)^N))$ such that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $(\mathbf{f}, \mathbf{h}, \mathbf{k}) \in X_{\mathcal{R},q}(\dot\Omega)$, $\mathbf{u} = \mathbf{B}(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}, \mathbf{k})$ is a unique solution to (2.3)*, and furthermore, for $l = 0, 1$,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{R,q}(\dot\Omega), L_q(\dot\Omega)^{\tilde N})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (R_\lambda\mathbf{B}(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \le \gamma_0, \tag{2.9}$$

with some positive constant γ_0. Here, we have set $\widetilde N = N^3 + N^2 + N$, $R_\lambda\mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$, and

$$F_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}, \mathbf{k}) = (\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}, \mathbf{h}, \nabla\mathbf{k}, \lambda^{1/2}\mathbf{k}, \mathbf{k}).$$

Remark 2.3. (1) The symbols H_1, H_2, H_3, H_4, H_5, H_6, and H_7 are variables corresponding to $\mathbf{f}$, $\nabla\mathbf{h}$, $\lambda^{1/2}\mathbf{h}$, $\mathbf{h}$, $\nabla\mathbf{k}$, $\lambda^{1/2}\mathbf{k}$, and $\mathbf{k}$, respectively. The norm of space $\mathcal{X}_{\mathcal{R},q}(\dot\Omega)$ is given by $\|(H_1, \ldots, H_7)\|_{\mathcal{X}_{\mathcal{R},q}(\dot\Omega)} = \|(H_1, H_2, H_3)\|_{L_q(\dot\Omega)} + \|H_4\|_{W^1_q(\dot\Omega)} + \|(H_5, H_6)\|_{L_q(\Omega_+)} + \|H_7\|_{W^1_q(\Omega_+)}$.

(2) If $\mathbf{u}$ satisfies (2.3) with $\mathbf{f} \in J_q(\Omega)$, $< [\![\mathbf{h}]\!], \mathbf{n} >= 0$ on Γ, $< \mathbf{k}, \mathbf{n}_+ >= 0$ on Γ_+, and $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, then $\operatorname{div} \mathbf{u} = 0$ in $\dot\Omega$. This fact can be obtained in the same manner as in Subsection 2.2 with $g = 0$. It then holds that $\mathbf{u}$ belongs to $J_q(\Omega)$ by Gauss's divergence theorem together with $[\![\mathbf{u}]\!] = 0$ on Γ, $\mathbf{u} = 0$ on Γ_-. Here, and subsequently, we can see $J_q(\Omega)$ as a closed subspace of $L_q(\dot\Omega)^N$, that is, $J_q(\Omega)$ is regarded as a Banach space endowed with $\|\cdot\|_{L_q(\dot\Omega)}$.

At this point, we introduce several propositions used throughout this paper. The following two propositions are fundamental properties of the $\mathcal{R}$-boundedness (cf. [12, Proposition 3.4], [12, Remark 3.2. (4)]).

Proposition 2.4. (1) *Let X, Y be Banach spaces, and let $\mathcal{T}$, $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$. Then, $\mathcal{T}+\mathcal{S} = \{T+S : T \in \mathcal{T},\ S \in \mathcal{S}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(X,Y)$ with $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S})$.*
(2) *Let X, Y, Z be Banach spaces, and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then, $\mathcal{S}\mathcal{T} = \{ST : S \in \mathcal{S},\ T \in \mathcal{T}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(X,Z)$ with $\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S})$.*

Proposition 2.5. *Let $1 \leq q < \infty$. Let $m(\lambda)$ be a bounded function defined on a subset Λ in the complex plane $\mathbf{C}$, and let $M_m(\lambda)$ be a multiplication operator defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(G)$ with open set G of $\mathbf{R}^N$. Then, $\mathcal{R}_{\mathcal{L}(L_q(G))}(\{M_m(\lambda) : \lambda \in \Lambda\}) \leq K_q^2\|m\|_{L_\infty(\Lambda)}$, where K_q is a positive constant in Khintchine's inequality* (*cf. also* [18, Theorem 2.4]).

The next proposition is used to estimate several terms arising from localization procedure in Sections 4, 5 (cf. [25, Lemma 2.4]).

Proposition 2.6. *Let $1 \leq q \leq r < \infty$ and $N < r < \infty$. Suppose that $\Omega_\pm$ are uniform $W_r^{2-1/r}$ domains. Then there exists a positive constant $C_{N,q,r}$ such that, for any $\sigma > 0$, $a \in L_r(\dot\Omega)$, and $b \in W_q^1(\dot\Omega)$, it holds the estimate:*

$$\|ab\|_{L_q(\dot\Omega)} \leq \sigma\|\nabla b\|_{L_q(\dot\Omega)} + C_{N,q,r}\left(\sigma^{-\frac{N}{r-N}}\|a\|_{L_r(\dot\Omega)}^{\frac{r}{r-N}} + \|a\|_{L_r(\dot\Omega)}\right)\|b\|_{L_q(\dot\Omega)}.$$

We devote the last part of this subsection to the proof of Theorem 1.6.

Proof of Theorem 1.6. We prove Theorem 1.6 under the assumption that Theorem 2.2 holds. The existence of $\mathbf{A}(\lambda)$, $\mathbf{P}(\lambda)$ follows from Theorem 2.2 and Proposition 2.4 as was discussed in Subsection 2.2.

Next, we show the uniqueness of solutions to (1.1). Let $\mathbf{u} \in W^2_q(\dot\Omega)^N \cap J_q(\Omega)$ and $\theta = \theta_1 + \theta_2 \in W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega)$ satisfy

$$\begin{cases} \lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},\theta) = 0 & \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u},\theta)\mathbf{n}]\!] = 0 & \text{on } \Gamma, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u},\theta)\mathbf{n}_+ = 0 & \text{on } \Gamma_+, \\ \mathbf{u} = 0 & \text{on } \Gamma_-. \end{cases} \tag{2.10}$$

We prove that $\mathbf{u} = 0$ in $\dot\Omega$, which leads to the uniqueness. To this end, it suffices to show that

$$(\rho\mathbf{u}, \psi)_{\dot\Omega} = 0 \quad \text{for any } \psi \in C^\infty_0(\dot\Omega)^N. \tag{2.11}$$

The assumption (b), stated in Theorem 1.6, allows us to choose a $\kappa \in \mathcal{W}^1_{q'}(\Omega)$ satisfying $(\rho^{-1}\nabla\kappa, \nabla\varphi)_{\dot\Omega} = (\psi, \nabla\varphi)_{\dot\Omega}$ for any $\varphi \in \mathcal{W}^1_q(\Omega)$. In addition, since the two-phase reduced Stokes resolvent equations (2.3) is solvable for $q' = q/(q-1)$, we have a solution $\mathbf{v} \in W^2_{q'}(\dot\Omega)^N$ to the equations:

$$\begin{cases} \lambda\mathbf{v} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{v},K(\mathbf{v})) = \psi - \rho^{-1}\nabla\kappa & \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{v},K(\mathbf{v}))\mathbf{n}]\!] = 0 & \text{on } \Gamma, \\ [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma, \\ \mathbf{T}(\mathbf{v},K(\mathbf{v}))\mathbf{n}_+ = 0 & \text{on } \Gamma_+, \\ \mathbf{v} = 0 & \text{on } \Gamma_-. \end{cases}$$

Then, $\psi - \rho^{-1}\nabla\kappa \in J_{q'}(\Omega)$ implies that $\mathbf{v} \in J_{q'}(\Omega)$ as was discussed in Remark 2.3 (2). Setting $K(\mathbf{v}) = w_1 + w_2 \in W^1_{q'}(\dot\Omega) + \mathcal{W}^1_{q'}(\Omega)$, we have, by Gauss's divergence theorem, $(\mathbf{u}, \nabla\kappa)_{\dot\Omega} = 0$, and $(\mathbf{u}, \nabla w_2)_{\dot\Omega} = 0$,

$$\begin{aligned} &(\rho\mathbf{u}, \psi)_{\dot\Omega} = (\rho\mathbf{u}, \lambda\mathbf{v} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{v}, w_1 + w_2 + \kappa))_{\dot\Omega} \\ &= \lambda(\rho\mathbf{u},\mathbf{v})_{\dot\Omega} + (\mathbf{D}(\mathbf{u}), \mu\mathbf{D}(\mathbf{v}))_{\dot\Omega} - (\mathbf{u}, [\![\mu\mathbf{D}(\mathbf{v})\mathbf{n}]\!])_\Gamma - (\mathbf{u}, \mu\mathbf{D}(\mathbf{v})\mathbf{n}_+)_{\Gamma_+} \\ &- (\operatorname{div}\mathbf{u}, w_1)_{\dot\Omega} + (\mathbf{u}, [\![w_1\mathbf{n}]\!])_\Gamma + (\mathbf{u}, w_1\mathbf{n}_+)_{\Gamma_+}. \end{aligned} \tag{2.12}$$

Noting that $[\![w_2]\!] = 0$ on Γ and $w_2 = 0$ on Γ_+, we see that $[\![\mu\mathbf{D}(\mathbf{v})\mathbf{n} - w_1\mathbf{n}]\!] = [\![\mu\mathbf{D}(\mathbf{v})\mathbf{n} - K(\mathbf{v})\mathbf{n}]\!] = 0$ on Γ and $\mu\mathbf{D}(\mathbf{v})\mathbf{n} - w_1\mathbf{n} = \mu\mathbf{D}(\mathbf{v}) - K(\mathbf{v})\mathbf{n} = 0$ on Γ_+. In addition, it holds that $\operatorname{div}\mathbf{u} = 0$ in $\dot\Omega$, since

$$0 = -(\mathbf{u}, \nabla\varphi)_{\dot\Omega} = (\operatorname{div}\mathbf{u}, \varphi)_{\dot\Omega} \quad \text{for any } \varphi \in C^\infty_0(\dot\Omega),$$

where we have used $\mathbf{u} \in J_q(\Omega)$ and the relation $C_0^\infty(\dot\Omega) \subset \mathcal{W}^1_{q'}(\Omega)$. Hence, (2.12) implies that

$$(\rho\mathbf{u}, \psi)_{\dot\Omega} = \lambda(\rho\mathbf{u}, \mathbf{v})_{\dot\Omega} + (\mathbf{D}(\mathbf{u}), \mu\mathbf{D}(\mathbf{v}))_{\dot\Omega}. \tag{2.13}$$

On the other hand, $\lambda\rho\mathbf{u} - \operatorname{Div}\mathbf{T}(\mathbf{u},\theta) = 0$ in $\dot\Omega$ by the first equation of (2.10), which, combined with Gauss's divergence theorem, furnishes that

$$\begin{aligned}0 = \lambda(\rho\mathbf{u}, \mathbf{v})_{\dot\Omega} &+ (\mu\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\dot\Omega} - ([\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}]\!], \mathbf{v})_\Gamma - (\mu\mathbf{D}(\mathbf{u})\mathbf{n}_+, \mathbf{v})_{\Gamma_+} \\ &- (\theta_1, \operatorname{div}\mathbf{v})_{\dot\Omega} + ([\![\theta_1\mathbf{n}]\!], \mathbf{v})_\Gamma + (\theta_1\mathbf{n}_+, \mathbf{v})_{\Gamma_+},\end{aligned}$$

because $(\nabla\theta_2, \mathbf{v})_{\dot\Omega} = 0$ by $\mathbf{v} \in J_{q'}(\Omega)$. Thus, $\lambda(\rho\mathbf{u}, \mathbf{v})_{\dot\Omega} + (\mu\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\dot\Omega} = 0$ in the same manner as we have obtained (2.13) from (2.12). The last identity combined with (2.13) implies (2.11). □

2.4. **Generation of analytic semigroup.** In this and the next subsection, we discuss time-dependent problems. We now consider the following initial-boundary value problem:

$$\begin{cases} \partial_t\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, K(\mathbf{u})) = 0 & \text{in } \dot\Omega \times (0,\infty), \\ [\![\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}]\!] = 0 & \text{on } \Gamma \times (0,\infty), \\ [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma \times (0,\infty), \\ \mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}_+ = 0 & \text{on } \Gamma_+ \times (0,\infty), \\ \mathbf{u} = 0 & \text{on } \Gamma_- \times (0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot\Omega. \end{cases} \tag{2.14}$$

To discuss the generation of analytic semigroup associated with (2.14), we formulate (2.14) in the semigroup setting. For this purpose, we introduce the Stokes operator $\mathcal{A}$ and its domain $\mathcal{D}_q(\mathcal{A})$ as follows:

$$\begin{aligned}\mathcal{D}_q(\mathcal{A}) = \{\mathbf{u} \in W_q^2(\dot\Omega)^N \cap J_q(\Omega) : [\![\mathcal{T}_\mathbf{n}(\mu\mathbf{D}(\mathbf{u})\mathbf{n})]\!] = 0 \quad \text{on } \Gamma, \\ [\![\mathbf{u}]\!] = 0 \quad \text{on } \Gamma, \quad \mathcal{T}_{\mathbf{n}_+}(\mu\mathbf{D}(\mathbf{u})\mathbf{n}_+) = 0 \quad \text{on } \Gamma_+, \quad \mathbf{u} = 0 \quad \text{on } \Gamma_-\}, \\ \mathcal{A}\mathbf{u} = \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, K(\mathbf{u})) \quad \text{for } \mathbf{u} \in \mathcal{D}_q(\mathcal{A}),\end{aligned} \tag{2.15}$$

where we have set $\mathcal{T}_\mathbf{n}\mathbf{f} = \mathbf{f} - < \mathbf{f}, \mathbf{n} > \mathbf{n}$ and $\mathcal{T}_{\mathbf{n}_+}\mathbf{f} = \mathbf{f} - < \mathbf{f}, \mathbf{n}_+ > \mathbf{n}_+$ that are the tangential parts of N-vector $\mathbf{f}$ with respect to $\mathbf{n}$ and $\mathbf{n}_+$, respectively. Then it is possible to rewrite (2.14) as follows:

$$\partial_t\mathbf{u} - \mathcal{A}\mathbf{u} = 0 \quad (t > 0), \quad \mathbf{u}|_{t=0} = \mathbf{u}_0.$$

By Theorem 2.2, the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\Sigma_{\varepsilon,\lambda_0}$. In addition, denoting the resolvent operator of $\mathcal{A}$ by $(\lambda - \mathcal{A})^{-1}$ and noting Remark 2.3 (2), we see that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $\mathbf{f} \in J_q(\Omega)$, $(\lambda - \mathcal{A})^{-1}\mathbf{f} =$

$\mathbf{B}(\lambda)(\mathbf{f}, 0, 0, 0, 0) \in J_q(\Omega)$. Since the $\mathcal{R}$-boundedness of $\mathbf{B}(\lambda)$ implies the usual boundedness, it holds that

$$\|(\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(J_q(\Omega))} \leq \frac{M_{\varepsilon,\lambda_0}}{|\lambda|} \quad (\lambda \in \Sigma_{\varepsilon,\lambda_0}),$$

with some positive constant $M_{\varepsilon,\lambda_0}$. By this resolvent estimate and the standard semigroup theory, we have the following theorem.

Theorem 2.7. *Let $1 < q < \infty$, $N < r < \infty$, and $\max(q', q) \leq r$ with $q' = q/(q-1)$. Let $\rho_\pm$ be positive constants. Suppose that the conditions* (a), (b), *and* (c) *stated in Theorem* 1.6 *hold. Then the Stokes operator $\mathcal{A}$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $J_q(\Omega)$, which is analytic.*

2.5. **Maximal L_p-L_q regularity.** Since the system (2.1) is linear, we consider the following two problems:

$$\begin{cases} \partial_t \mathbf{u} - \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{u}, \theta) = 0, \quad \operatorname{div} \mathbf{u} = 0 & \text{in } \dot{\Omega} \times (0, \infty), \\ [\![\mathbf{T}(\mathbf{u}, \theta)\mathbf{n}]\!] = 0, \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma \times (0, \infty), \\ \mathbf{T}(\mathbf{u}, \theta)\mathbf{n}_+ = 0 & \text{on } \Gamma_+ \times (0, \infty), \\ \mathbf{u} = 0 & \text{on } \Gamma_- \times (0, \infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot{\Omega}, \end{cases} \tag{2.16}$$

$$\begin{cases} \partial_t \mathbf{u} - \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{u}, \theta) = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = g & \text{in } \dot{\Omega} \times (0, \infty), \\ [\![\mathbf{T}(\mathbf{u}, \theta)\mathbf{n}]\!] = [\![\mathbf{h}]\!], \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma \times (0, \infty), \\ \mathbf{T}(\mathbf{u}, \theta)\mathbf{n}_+ = \mathbf{k} & \text{on } \Gamma_+ \times (0, \infty), \\ \mathbf{u} = 0 & \text{on } \Gamma_- \times (0, \infty), \\ \mathbf{u}|_{t=0} = 0 & \text{in } \dot{\Omega}. \end{cases} \tag{2.17}$$

To state maximal regularity theorems for (2.16) and (2.17), we introduce several function spaces. For a Banach space X, we denote the usual Lebesgue space and Sobolev space of X-valued functions defined on time interval I by $L_p(I, X)$ and $W^m_p(I, X)$ with $m \in \mathbf{N}$, and their associated norms by $\|\cdot\|_{L_p(I,X)}$ and $\|\cdot\|_{W^m_p(I,X)}$, respectively. We set for $\gamma > 0$

$$\begin{aligned} &L_{p,\gamma}(I, X) = \{f : I \to X : e^{-\gamma t} f \in L_p(I, X)\}, \\ &L_{p,0,\gamma}(\mathbf{R}, X) = \{f \in L_{p,\gamma}(\mathbf{R}, X) : f(t) = 0 \ \text{ for } t < 0\}, \\ &W^m_{p,\gamma}(I, X) = \{f \in L_{p,\gamma}(I, X) : e^{-\gamma t}\partial_t^j f(t) \in L_p(I, X) \ \ (j = 1, \ldots, m)\}, \\ &W^m_{p,0,\gamma}(\mathbf{R}, X) = W^m_{p,\gamma}(\mathbf{R}, X) \cap L_{p,0,\gamma}(\mathbf{R}, X). \end{aligned}$$

Let $\mathcal{L}$, $\mathcal{L}^{-1}$, $\mathcal{F}$, and $\mathcal{F}^{-1}$ denote the Laplace transform, the Laplace inverse transform, the Fourier transform, and the Fourier inverse transform, which are denoted by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\,dt, \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t} g(\lambda)\,d\tau \quad (\lambda = \gamma + i\tau),$$
$$\mathcal{F}[f](\tau) = \int_{-\infty}^{\infty} e^{-i\tau t} f(t)\,dt, \quad \mathcal{F}^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t} g(\tau)\,d\tau.$$

Note that $\mathcal{L}[f](\lambda) = \mathcal{F}[e^{-\gamma t} f(t)](\tau)$ and $\mathcal{L}^{-1}[g](t) = e^{\gamma t}\mathcal{F}^{-1}[g(\gamma+i\tau)](t)$. For any real number $s \geq 0$, let $(\Lambda^s_\gamma f)(t) = \mathcal{L}^{-1}[\lambda^s \mathcal{L}[f]](t)$ and set $H^s_{p,\gamma}(\mathbf{R}, X)$ as the Bessel potential space of order s defined by

$$H^s_{p,\gamma}(\mathbf{R}, X) = \{f \in L_{p,\gamma}(\mathbf{R}, X) : e^{-\gamma t}(\Lambda^s_\gamma f)(t) \in L_p(\mathbf{R}, X)\}.$$

We also set $H^s_{p,0,\gamma}(\mathbf{R}, X) = \{f \in H^s_{p,\gamma}(\mathbf{R}, X) : f(t) = 0 \text{ for } t < 0\}$. For solutions of problems (2.16) and (2.17), $W^{2,1}_{q,p,\gamma}(\dot\Omega \times (0,\infty))$ and $W^{2,1}_{q,p,0,\gamma}(\dot\Omega \times \mathbf{R})$ are defined by

$$W^{2,1}_{q,p,\gamma}(\dot\Omega \times (0,\infty)) = W^1_{p,\gamma}((0,\infty), L_q(\dot\Omega)^N) \cap L_{p,\gamma}((0,\infty), W^2_q(\dot\Omega)^N),$$
$$W^{2,1}_{q,p,0,\gamma}(\dot\Omega \times \mathbf{R}) = W^1_{p,0,\gamma}(\mathbf{R}, L_q(\dot\Omega)^N) \cap L_{p,0,\gamma}(\mathbf{R}, W^2_q(\dot\Omega)^N).$$

First, we discuss a maximal L_p-L_q regularity theorem for (2.16). Setting $\mathbf{u}(t) = T(t)\mathbf{u}_0$ and $\theta(t) = K(\mathbf{u}(t))$, we see that $\operatorname{div}\mathbf{u}(t) = 0$ in $\dot\Omega$ for $t > 0$ by $\mathbf{u}(t) \in J_q(\Omega)$, and thus $\mathbf{u}(t)$ and $\theta(t)$ satisfy (2.16). Since $\{T(t)\}_{t\geq 0}$ is analytic, we have, for some $\lambda_0 \geq 1$ and for any $t > 0$,

$$\begin{aligned}
\|T(t)\mathbf{u}_0\|_{J_q(\Omega)} &\leq C_{q,\lambda_0} e^{\lambda_0 t}\|\mathbf{u}_0\|_{J_q(\Omega)} && \text{for } \mathbf{u}_0 \in J_q(\Omega),\\
\|\partial_t T(t)\mathbf{u}_0\|_{J_q(\Omega)} &\leq C_{q,\lambda_0} t^{-1} e^{\lambda_0 t}\|\mathbf{u}_0\|_{J_q(\Omega)} && \text{for } \mathbf{u}_0 \in J_q(\Omega),\\
\|\partial_t T(t)\mathbf{u}_0\|_{J_q(\Omega)} &\leq C_{q,\lambda_0} e^{\lambda_0 t}\|\mathbf{u}_0\|_{\mathcal{D}_q(\mathcal{A})} && \text{for } \mathbf{u}_0 \in \mathcal{D}_q(\mathcal{A}),
\end{aligned}$$

with some positive constant C_{q,λ_0}. We then obtain, in the same manner as in [27, Theorem 3.9],

$$\|e^{-2\lambda_0 t}(\partial_t\mathbf{u}, \mathbf{u}, \nabla\mathbf{u}, \nabla^2\mathbf{u})\|_{L_p((0,\infty),L_q(\dot\Omega))} \leq C_{p,q,\lambda_0}\|\mathbf{u}_0\|_{\mathcal{D}^{2(1-1/p)}_{q,p}(\dot\Omega)},$$

for some positive constant C_{p,q,λ_0} with $1 < p, q < \infty$, where we have set $\mathcal{D}^{2(1-1/p)}_{q,p}(\dot\Omega) = (J_q(\Omega), \mathcal{D}_q(\mathcal{A}))_{1-1/p,p}$ with real interpolation functor $(\cdot,\cdot)_{\theta,p}$ $(0 < \theta < 1,\ 1 < p < \infty)$. Then, the following theorem holds.

Theorem 2.8. *Let* $1 < p, q < \infty$, $N < r < \infty$, *and* $\max(q, q') \leq r$ *with* $q' = q/(q-1)$. *Let* $\rho_\pm$ *be positive constants. Suppose that the conditions* (a), (b), (c) *stated in Theorem* 1.6 *hold. Then, we have the following assertions:*

(1) *There exists a constant* $\gamma_0 \geq 1$ *such that, for any* $\mathbf{u}_0 \in \mathcal{D}^{2(1-1/p)}_{q,p}(\dot\Omega)$, *the problem* (2.16) *admits a unique solution* $(\mathbf{u}, \theta) \in W^{2,1}_{q,p,\gamma_0}(\dot\Omega \times (0,\infty)) \times L_{p,\gamma_0}((0,\infty), W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega))$, *which satisfies*

$$\|e^{-\gamma_0 t}(\partial_t \mathbf{u}, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u})\|_{L_p((0,\infty), L_q(\dot\Omega))} + \|e^{-\gamma_0 t}\nabla\theta\|_{L_p((0,\infty), L_q(\dot\Omega))}$$
$$\leq C_{p,q,\gamma_0}\|\mathbf{u}_0\|_{\mathcal{D}^{2(1-1/p)}_{q,p}(\dot\Omega)},$$

with some positive constant C_{p,q,γ_0}.

(2) *There exists a positive constant* $\gamma_0 \geq 1$ *such that, for any*

$$\mathbf{f} \in L_{p,0,\gamma_0}(\mathbf{R}, L_q(\dot\Omega)^N),$$
$$g \in H^{1/2}_{p,0,\gamma_0}(\mathbf{R}, L_q(\dot\Omega)^N) \cap L_{p,0,\gamma_0}(\mathbf{R}, W^1_q(\dot\Omega) \cap \mathbf{W}^{-1}_q(\Omega))),$$
$$\mathbf{h} \in H^{1/2}_{p,0,\gamma_0}(\mathbf{R}, L_q(\dot\Omega)^N) \cap L_{p,0,\gamma_0}(\mathbf{R}, W^1_q(\dot\Omega)^N),$$
$$\mathbf{k} \in H^{1/2}_{p,0,\gamma_0}(\mathbf{R}, L_q(\Omega_+)^N) \cap W^1_{p,0,\gamma_0}(\mathbf{R}, L_q(\Omega_+)^N),$$

and for any representative $\mathfrak{g} \in W^1_{p,0,\gamma_0}(\mathbf{R}, L_q(\dot\Omega)^N)$ *of* $\mathcal{G}(g)$, *the problem* (2.17) *has a unique solution* $(\mathbf{u}, \theta) \in W^{2,1}_{q,p,0,\gamma_0}(\dot\Omega \times \mathbf{R}) \times L_{p,0,\gamma_0}(\mathbf{R}, W^1_q(\dot\Omega) + \mathcal{W}^1_q(\Omega))$, *which possesses the estimate:*

$$\|e^{-\gamma_0 t}(\partial_t \mathbf{u}, \mathbf{u}, \Lambda^{1/2}_{\gamma_0}\nabla \mathbf{u}, \nabla^2 \mathbf{u})\|_{L_p(\mathbf{R}, L_q(\dot\Omega))} + \|e^{-\gamma_0 t}\nabla\theta\|_{L_p(\mathbf{R}, L_q(\dot\Omega))} \tag{2.18}$$
$$\leq C_{p,q,\gamma_0}\mathcal{N}_{p,q,\gamma_0}(\mathbf{f}, g, \mathfrak{g}, \mathbf{h}, \mathbf{k}),$$

for some positive constant C_{p,q,γ_0} *with*

$$\mathcal{N}_{p,q,\gamma_0}(\mathbf{f}, g, \mathfrak{g}, \mathbf{h}, \mathbf{k}) = \|e^{-\gamma_0 t}(\mathbf{f}, \nabla g, \Lambda^{1/2}_{\gamma_0} g, \partial_t \mathfrak{g}, \nabla \mathbf{h}, \Lambda^{1/2}_{\gamma_0}\mathbf{h})\|_{L_p(\mathbf{R}, L_q(\dot\Omega))}$$
$$+ \|e^{-\gamma_0 t}(g, \mathbf{h})\|_{L_p(\mathbf{R}, W^1_q(\dot\Omega))} + \|e^{-\gamma_0 t}(\nabla \mathbf{k}, \Lambda^{1/2}_{\gamma_0}\mathbf{k})\|_{L_p(\mathbf{R}, L_q(\Omega_+))}$$
$$+ \|e^{-\gamma_0 t}\mathbf{k}\|_{L_p(\mathbf{R}, W^1_q(\Omega_+))}.$$

In addition, if $g = 0$, $\mathbf{h} = 0$, *and* $\mathbf{k} = 0$, *then*

$$\gamma\|e^{-\gamma t}\mathbf{u}\|_{L_p(\mathbf{R}, L_q(\dot\Omega))} \leq C_{p,q,\gamma_0}\|e^{-\gamma_0 t}\mathbf{f}\|_{L_p(\mathbf{R}, L_q(\dot\Omega))} \quad \text{for any } \gamma \geq \gamma_0. \tag{2.19}$$

Proof. The assertion (1) was already proved above. The estimates (2.18), (2.19) follows from Weis's operator valued Fourier multiplier theorem (cf. [31, Theorem 3.4]) together with Theorem 1.6 and Propositions 2.4, 2.5. Then, similarly to [24, Section 7], we see that $\mathbf{u}(t) = 0$ and $\theta(t) = 0$ for $t < 0$ and that the uniqueness holds. □

3. Two-phase reduced Stokes resolvent equations in $\dot{\mathbf{R}}^N$

In this section, we discuss $\mathcal{R}$-bounded solution operator families to the two-phase reduced Stokes resolvent equations with an interface condition in $\dot{\mathbf{R}}^N = \mathbf{R}^N_+ \cup \mathbf{R}^N_-$, that is, we consider the following resolvent problem with resolvent parameter λ varying in $\Sigma_\varepsilon = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg\lambda| < \pi - \varepsilon\}$:

$$\begin{cases} \lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, K_I(\mathbf{u})) = \mathbf{f} & \text{in } \dot{\mathbf{R}}^N, \\ [\![\mathbf{T}(\mathbf{u}, K_I(\mathbf{u}))\mathbf{n}_0]\!] = [\![\mathbf{h}]\!] & \text{on } \mathbf{R}^N_0, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^N_0, \end{cases} \tag{3.1}$$

where $\mathbf{n}_0 = (0, \dots, 0, -1)^T$ and $\mathbf{T}(\mathbf{u}, K_I(\mathbf{u})) = \mu\mathbf{D}(\mathbf{u}) - K_I(\mathbf{u})\mathbf{I}$. Here, $\rho = \rho_+\chi_{\mathbf{R}^N_+} + \rho_-\chi_{\mathbf{R}^N_-}$ for positive constants $\rho_\pm$, and suppose that

(d) viscosity coefficient μ is given by $\mu = \mu_+\chi_{\mathbf{R}^N_+} + \mu_-\chi_{\mathbf{R}^N_-}$ for positive constants $\mu_\pm$ satisfying $\mu_{\pm1} \le \mu_\pm \le \mu_{\pm2}$, respectively, where $\mu_{\pm1}$ and $\mu_{\pm2}$ are the same constants as in Theorem 1.6.

Furthermore, for $1 < q < \infty$ and $q' = q/(q-1)$, let $K_I(\mathbf{u})$ be defined by $K_I(\mathbf{u}) = \mathcal{K}(\alpha, \beta)$ with

$$\alpha = \rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u})) - \nabla\operatorname{div}\mathbf{u}, \quad \beta = < [\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}_0]\!], \mathbf{n}_0 > - [\![\operatorname{div}\mathbf{u}]\!],$$

for $\mathbf{u} \in W^2_q(\dot{\mathbf{R}}^N)^N$, where $\mathcal{K}(\alpha, \beta)$ is given in Remark 1.5 (3) with $\dot\Omega = \dot{\mathbf{R}}^N$, i.e., $K_I(\mathbf{u})$ is the unique solution to

$$(\rho^{-1}\nabla K_I(\mathbf{u}), \nabla\varphi)_{\dot{\mathbf{R}}^N} = (\rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u})) - \nabla\operatorname{div}\mathbf{u}, \nabla\varphi)_{\dot{\mathbf{R}}^N}, \tag{3.2}$$

$$[\![K_I(\mathbf{u})]\!] = < [\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}_0]\!], \mathbf{n}_0 > - [\![\operatorname{div}\mathbf{u}]\!] \quad \text{on } \mathbf{R}^N_0, \tag{3.3}$$

for all $\varphi \in \widehat{W}^1_{q'}(\mathbf{R}^N)$ and satisfies $\|\nabla K_I(\mathbf{u})\|_{L_q(\dot{\mathbf{R}}^N)} \le \gamma_0\|\nabla\mathbf{u}\|_{W^1_q(\dot{\mathbf{R}}^N)}$. Here, and hereafter, γ_0 denotes a generic constant depending solely on N, q, ρ_+, ρ_-, μ_{+1}, μ_{+2}, μ_{-1}, and μ_{-2}.

We prove the following theorem in this section.

Theorem 3.1. *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, and $\rho_\pm$ be positive constants. Suppose that the condition* (d) *holds. For any open set G of $\mathbf{R}^N$, let $Y_{\mathcal{R},q}(G)$ and $\mathcal{Y}_{\mathcal{R},q}(G)$ be defined as*

$$Y_{\mathcal{R},q}(G) = \{(\mathbf{f}, \mathbf{h}) : \mathbf{f} \in L_q(G)^N,\ \mathbf{h} \in W^1_q(G)^N\},$$

$$\mathcal{Y}_{\mathcal{R},q}(G) = \{(H_1, H_2, H_3) : H_1, H_3 \in L_q(G)^N,\ H_2 \in L_q(G)^{N^2}\}.$$

Then, there is an operator family $\mathbf{S}_I(\lambda) \in \operatorname{Hol}(\Sigma_\varepsilon, \mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), W^2_q(\dot{\mathbf{R}}^N)^N))$ such that, for any $\lambda \in \Sigma_\varepsilon$ and $(\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$, $\mathbf{u} = \mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})$ is a *unique solution to the problem* (3.1)*, and furthermore,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N),L_q(\dot{\mathbf{R}}^N)^{\widetilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l(R_\lambda \mathbf{S}_I(\lambda)) : \lambda\in\Sigma_\varepsilon\Big\}\Big) \le \gamma_1 \quad (l=0,1).$$

Here, and subsequently, we set $\widetilde{N} = N^3+N^2+N$, $R_\lambda\mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$, $G_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h}) = (\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})$ *and* γ_1 *denotes a constant depending solely on* N*,* q*,* ε*,* ρ_+*,* ρ_-*,* μ_{+1}*,* μ_{+2}*,* μ_{-1}*, and* μ_{-2}*.*

In view of Subsection 2.1, it is sufficient to consider the two-phase Stokes resolvent equations in $\dot{\mathbf{R}}^N$:

$$\begin{cases} \lambda\rho\mathbf{u} - \mathrm{Div}(\mu\mathbf{D}(\mathbf{u})) + \nabla\theta = \rho\mathbf{f} & \text{in } \dot{\mathbf{R}}^N, \\ \mathrm{div}\,\mathbf{u} = g & \text{in } \dot{\mathbf{R}}^N, \\ [\![(\mu\mathbf{D}(\mathbf{u}) - \theta\mathbf{I})\mathbf{n}_0]\!] = [\![\mathbf{h}]\!] & \text{on } \mathbf{R}^N_0, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \mathbf{R}^N_0. \end{cases} \tag{3.4}$$

Here, the Fourier transform $\mathcal{F}$ and its inverse formula $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbf{R}^N} e^{-ix\cdot\xi}f(x)\,dx, \quad \mathcal{F}^{-1}[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbf{R}^N} e^{ix\cdot\xi}g(\xi)\,d\xi, \tag{3.5}$$

respectively. We first consider the divergence equation: $\mathrm{div}\,\mathbf{u} = g$ in $\dot{\mathbf{R}}^N$.

Lemma 3.2. *Let* $1 < q < \infty$*. For* $g \in W^1_q(\dot{\mathbf{R}}^N) \cap \mathbf{W}^{-1}_q(\mathbf{R}^N)$*, we set*

$$V(g) = (V_1(g), \ldots, V_N(g))^T, \quad V_j(g) = -\mathcal{F}^{-1}\Big[\frac{i\xi_j}{|\xi|^2}\mathcal{F}[g](\xi)\Big](x), \tag{3.6}$$

for $j = 1, \ldots, N$*. Then,* $V(g) \in W^1_q(\mathbf{R}^N)^N \cap W^2_q(\dot{\mathbf{R}}^N)^N$ *and* $\mathbf{u} = V(g)$ *solves the divergence equation:* $\mathrm{div}\,\mathbf{u} = g$ *in* $\dot{\mathbf{R}}^N$*. In addition, there are operators*

$$V^1 \in \mathcal{L}(L_q(\dot{\mathbf{R}}^N)^N, L_q(\dot{\mathbf{R}}^N)^{N^3}), \quad V^2 \in \mathcal{L}(L_q(\dot{\mathbf{R}}^N), L_q(\dot{\mathbf{R}}^N)^{N^2}),$$
$$V^3 \in \mathcal{L}(\widehat{W}^{-1}_q(\mathbf{R}^N), L_q(\dot{\mathbf{R}}^N)^N),$$

such that $R_\lambda V(g) = (V^1(\nabla g), V^2(\lambda^{1/2}g), V^3(\lambda g))$*, where the dual space of* $\widehat{W}^1_{q'}(\mathbf{R}^N)$*,* $q' = q/(q-1)$*, is written by* $\widehat{W}^{-1}_q(\mathbf{R}^N)$ *with norm* $\|\cdot\|_{\widehat{W}^{-1}_q(\mathbf{R}^N)}$*.*

Proof. It is clear that $\mathbf{u} = V(g)$ solves the divergence equation: $\mathrm{div}\,\mathbf{u} = g$ in $\dot{\mathbf{R}}^N$ and that by the Fourier multiplier theorem of Mikhlin (cf. [19, Appendix, Theorem 2])

$$\|\nabla V(g)\|_{L_q(\mathbf{R}^N)} \le \gamma_0\|g\|_{L_q(\mathbf{R}^N)}, \quad \|\partial_k\nabla V(g)\|_{L_q(\mathbf{R}^N)} \le \gamma_0\|\partial_k g\|_{L_q(\mathbf{R}^N)},$$

for $k = 1, \dots, N-1$. Since $\operatorname{div} V(g) = g$ in $\dot{\mathbf{R}}^N$, it holds that $\partial_N^2 V(g) = \partial_N g - \partial_N \sum_{k=1}^{N-1} \partial_k V(g)$ in $\dot{\mathbf{R}}^N$, which, combined with the last inequalities, furnishes that $\|\partial_N^2 V(g)\|_{L_q(\dot{\mathbf{R}}^N)} \leq \gamma_0 \|\nabla g\|_{L_q(\dot{\mathbf{R}}^N)}$.

Next, we estimate $V(g)$. Let $\varphi \in C_0^\infty(\mathbf{R}^N)^N$, and then $(V(g), \varphi)_{\mathbf{R}^N} = -(g, \mathcal{F}[|\xi|^{-2} i\xi \cdot \mathcal{F}^{-1}[\varphi](\xi)])_{\mathbf{R}^N}$. The Fourier multiplier theorem yields that $|(V(g), \varphi)_{\mathbf{R}^N}| \leq \gamma_0 \|g\|_{\widehat{W}_q^{-1}(\mathbf{R}^N)} \|\varphi\|_{L_{q'}(\mathbf{R}^N)}$, which furnishes $\|V(g)\|_{L_q(\mathbf{R}^N)} \leq \gamma_0 \|g\|_{\widehat{W}_q^{-1}(\mathbf{R}^N)}$. We thus see that $V(g) \in W_q^1(\mathbf{R}^N)^N \cap W_q^2(\dot{\mathbf{R}}^N)^N$ and the existence of operators V^i $(i = 1, 2, 3)$. This completes the proof. □

Note that $[\![V(g)]\!] = 0$ on $\mathbf{R}_0^N$ since $V(g) \in W_q^1(\mathbf{R}^N)^N$ by Lemma 3.2. Setting $\mathbf{u} = V(g) + \mathbf{v}$ in (3.4) and noting $\operatorname{Div}(\mu \mathbf{D}(\mathbf{v})) = \mu \Delta \mathbf{v}$ by the condition (d) and by $\operatorname{div} \mathbf{v} = 0$ in $\dot{\mathbf{R}}^N$, we have

$$\begin{cases} \rho\lambda \mathbf{v} - \mu \Delta \mathbf{v} + \nabla \theta = \widetilde{\mathbf{f}} & \text{in } \dot{\mathbf{R}}^N, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \dot{\mathbf{R}}^N, \\ [\![(\mu \mathbf{D}(\mathbf{v}) - \theta \mathbf{I})\mathbf{n}_0]\!] = [\![\widetilde{\mathbf{h}}]\!] & \text{on } \mathbf{R}_0^N, \\ [\![\mathbf{v}]\!] = 0 & \text{on } \mathbf{R}_0^N, \end{cases} \tag{3.7}$$

where $\widetilde{\mathbf{f}} = \rho \mathbf{f} - \rho\lambda V(g) + \operatorname{Div}(\mu \mathbf{D}(V(g)))$ and $\widetilde{\mathbf{h}} = \mathbf{h} - \mu \mathbf{D}(V(g))\mathbf{n}_0$.

The following lemma was essentially proved in [28, Theorems 1.1, 1.2].

Lemma 3.3. *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, and $\rho_\pm$ be positive constants. Suppose that the condition* (d) *holds. Then there exists an operator family $\mathcal{S}_I(\lambda) \in \operatorname{Hol}(\Sigma_\varepsilon, \mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), W_q^2(\dot{\mathbf{R}}^N)^N))$ such that, for any $\lambda \in \Sigma_\varepsilon$ and $(\widetilde{\mathbf{f}}, \widetilde{\mathbf{h}}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$, $\mathbf{v} = \mathcal{S}_I(\lambda) G_{\mathcal{R},\lambda}(\widetilde{\mathbf{f}}, \widetilde{\mathbf{h}})$ is a unique solution to the problem* (3.7) *with some pressure term θ. In addition, for $l = 0, 1$,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), L_q(\dot{\mathbf{R}}^N)^{\tilde{N}})} \Big(\Big\{ \Big(\lambda \frac{d}{d\lambda} \Big)^l (R_\lambda \mathcal{S}_I(\lambda)) : \lambda \in \Sigma_\varepsilon \Big\} \Big) \leq \gamma_1.$$

Since $R_\lambda V(g) = (V^1(\nabla g), V^2(\lambda^{1/2} g), V^3(\lambda g))$ by Lemma 3.2, we have the following lemma by combining Lemma 3.3 with Lemma 3.2 and by setting

$$Y_q = \big\{ (\mathbf{f}, g, \mathbf{h}) : \mathbf{f} \in L_q(\dot{\mathbf{R}}^N)^N, g \in W_q^1(\dot{\mathbf{R}}^N) \cap \mathbf{W}_q^{-1}(\mathbf{R}^N), \mathbf{h} \in W_q^1(\dot{\mathbf{R}}^N)^N \big\},$$
$$\mathcal{Y}_q = \big\{ (F_1, \dots, F_6) : F_1, F_4, F_6 \in L_q(\dot{\mathbf{R}}^N)^N, F_2 \in L_q(\dot{\mathbf{R}}^N)^{N^3},$$
$$F_3, F_5 \in L_q(\dot{\mathbf{R}}^N)^{N^2} \big\}, \quad G_\lambda(\mathbf{f}, g, \mathbf{h}) = (\mathbf{f}, R_\lambda V(g), \nabla \mathbf{h}, \lambda^{1/2} \mathbf{h}).$$

Lemma 3.4. *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, and constants $\rho_\pm > 0$, and let V be the same operator as in Lemma* 3.2*. Suppose that the condition* (d) *holds. Then, there is an operator family* $\mathcal{T}_I(\lambda) \in \mathrm{Hol}\,(\Sigma_\varepsilon, \mathcal{L}(\mathcal{Y}_q, W^2_q(\dot{\mathbf{R}}^N)^N))$ *such that* $\mathbf{u} = V(g) + \mathcal{T}_I(\lambda)G_\lambda(\mathbf{f}, g, \mathbf{h})$ *is a unique solution to the problem* (3.4) *with some pressure* θ *for* $\lambda \in \Sigma_\varepsilon$ *and* $(\mathbf{f}, g, \mathbf{h}) \in Y_q$. *In addition,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q, L_q(\dot{\mathbf{R}}^N)^{\tilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l \Big(R_\lambda \mathcal{T}_I(\lambda)\Big) : \lambda \in \Sigma_\varepsilon\Big\}\Big) \leq \gamma_1 \quad (l = 0, 1).$$

Proof of Theorem 3.1. Let $1 < q < \infty$ and $q' = q/(q-1)$. According to what was pointed out in Subsection 2.1, we consider, as an auxiliary problem, the following weak problem: for all $\varphi \in W^1_{q'}(\mathbf{R}^N)$,

$$\lambda(g, \varphi)_{\dot{\mathbf{R}}^N} + (\nabla g, \nabla\varphi)_{\dot{\mathbf{R}}^N} = -(\mathbf{f}, \nabla\varphi)_{\dot{\mathbf{R}}^N}, \;\; [\![g]\!] =< [\![\mathbf{h}]\!], \mathbf{n}_0 > \text{ on } \mathbf{R}^N_0. \quad (3.8)$$

Concerning this weak problem, we have

Proposition 3.5. *Let* $0 < \varepsilon < \pi/2$ *and* $1 < q < \infty$. *Suppose that* V *is the same operator as in Lemma* 3.2. *Then, for any* $\lambda \in \Sigma_\varepsilon$ *and* $(\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$, *the problem* (3.8) *admits a unique solution* $g \in W^1_q(\dot{\mathbf{R}}^N) \cap \mathbf{W}^{-1}_q(\mathbf{R}^N)$. *In addition, there is* $\mathbf{V}(\lambda) \in \mathrm{Hol}\,(\Sigma_\varepsilon, \mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), W^2_q(\dot{\mathbf{R}}^N)^N))$ *such that, for* $l = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), L_q(\dot{\mathbf{R}}^N)^{\tilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l \Big(R_\lambda \mathbf{V}(\lambda)\Big) : \lambda \in \Sigma_\varepsilon\Big\}\Big) \leq \gamma_1, \qquad (3.9)$$

and $V(g) = \mathbf{V}(\lambda)(\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})$ *for any* $(\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$, *where* g *is the solution to* (3.8).

Proof. We only show the existence of the $\mathcal{R}$-bounded solution operator family $\mathbf{V}(\lambda)$, since the unique solvability of the weak problem (3.8) was already mentioned in Proposition 2.1.

It suffices to consider the case $\mathbf{f} \in C^\infty_0(\dot{\mathbf{R}}^N)^N$ in what follows, since $C^\infty_0(\dot{\mathbf{R}}^N)$ is dense in $L_q(\dot{\mathbf{R}}^N)$. Then, the g satisfying (3.8) is given by $g = \varphi + \psi$ with

$$(\lambda - \Delta)\varphi = \mathrm{div}\,\mathbf{f} \quad \text{in } \mathbf{R}^N, \qquad \begin{cases} (\lambda - \Delta)\psi = 0 & \text{in } \dot{\mathbf{R}}^N, \\ [\![\psi]\!] = [\![h]\!], \quad [\![\dfrac{\partial\psi}{\partial\mathbf{n}_0}]\!] = 0 & \text{on } \mathbf{R}^N_0, \end{cases}$$

where $h =< \mathbf{h}, \mathbf{n}_0 >$ and $\partial\psi/\partial\mathbf{n}_0 = \mathbf{n}_0 \cdot \nabla\psi = -\partial_N\psi$.

Step 1: Solution formulas. We give the exact solution formulas of φ, ψ. By using (3.5), we have

$$\varphi = \mathcal{F}^{-1}\Big[\frac{\mathcal{F}[\mathrm{div}\,\mathbf{f}](\xi)}{\lambda + |\xi|^2}\Big](x) = \mathcal{F}^{-1}\Big[\frac{< i\xi, \mathcal{F}[\mathbf{f}](\xi) >}{\lambda + |\xi|^2}\Big](x). \qquad (3.10)$$

On the other hand, we rewrite the system for ψ as follows:

$$\begin{cases} (\lambda-\Delta)\psi_\pm = 0 & \text{in } \mathbf{R}^N_\pm, \\ \psi_+ - \psi_- = [\![h]\!] & \text{on } \mathbf{R}^N_0, \\ \partial_N\psi_+ - \partial_N\psi_- = 0 & \text{on } \mathbf{R}^N_0, \end{cases} \tag{3.11}$$

where we have set $\psi_\pm = \psi\chi_{\mathbf{R}^N_\pm}$. Let $\widehat{f}(\xi', x_N)$ and $\mathcal{F}^{-1}_{\xi'}[g(\xi', x_N)](x')$ be the partial Fourier transform with respect to x' and its inverse defined by

$$\widehat{f}(\xi', x_N) = \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)\,dx',$$
$$\mathcal{F}^{-1}_{\xi'}[g(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', x_N)\,d\xi'.$$

Applying the partial Fourier transform to (3.11) furnishes that

$$\begin{cases} \left\{\partial_N^2 - (\lambda+|\xi'|^2)\right\}\widehat{\psi}_\pm(\xi', x_N) = 0, \quad \pm x_N > 0, \\ \widehat{\psi}_+(\xi', 0) - \widehat{\psi}_-(\xi', 0) = [\![\widehat{h}]\!](\xi', 0), \\ (\partial_N\widehat{\psi}_+)(\xi', 0) - (\partial_N\widehat{\psi}_-)(\xi', 0) = 0. \end{cases}$$

Solving this system as ordinary differential equations with respect to x_N and setting $B = \sqrt{\lambda + A^2}$ for $\lambda \in \Sigma_\varepsilon$ and $A = |\xi'|$, we obtain $\widehat{\psi}_\pm(\xi', x_N) = \pm(1/2)[\![\widehat{h}]\!](\xi', 0)e^{\mp Bx_N}$ $(\pm x_N > 0)$, which implies that

$$\psi_\pm = \psi_\pm(x', x_N) = \pm\frac{1}{2}\mathcal{F}^{-1}_{\xi'}\left[[\![\widehat{h}]\!](\xi', 0)e^{\mp Bx_N}\right](x') \quad (\pm x_N > 0), \tag{3.12}$$

solves the problem (3.11). Hence, $\psi = \psi_+\chi_{\mathbf{R}^N_+} + \psi_-\chi_{\mathbf{R}^N_-}$.

Step 2: Construction of $\mathcal{R}$-bounded solution operator families. Since $V(\varphi+\psi) = V(\varphi)+V(\psi)$, we consider $V(\varphi)$, $V(\psi)$ one by one. First, we construct a $\mathcal{R}$-bounded solution operator family for $V(\varphi)$. By (3.6), (3.10),

$$V(\varphi) = \mathcal{F}^{-1}\Big[\frac{\xi < \xi, \mathcal{F}[\mathbf{f}](\xi) >}{|\xi|^2(\lambda+|\xi|^2)}\Big](x) =: \mathbf{V}^1(\lambda)\mathbf{f}.$$

By [14, Lemma 3.1, Theorem 3.3], we have, for $l = 0, 1$,

$$\mathbf{V}^1(\lambda) \in \mathrm{Hol}\,(\Sigma_\varepsilon, \mathcal{L}(L_q(\mathbf{R}^N)^N, W^2_q(\mathbf{R}^N)^N)),$$
$$\mathcal{R}_{\mathcal{L}(L_q(\mathbf{R}^N)^N, L_q(\mathbf{R}^N)^{\tilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l\big(R_\lambda\mathbf{V}^1(\lambda)\big) : \lambda \in \Sigma_\varepsilon\Big\}\Big) \le \gamma_1.$$

Next, we consider $V(\psi)$. By (3.6), we have, for $j = 1, \ldots, N-1$,

$$\widehat{V_j(\psi)} = -\int_{-\infty}^{\infty} i\xi_j \widehat{\psi}(\xi', y_N)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i(x_N-y_N)\xi_N}}{|\xi|^2}\,d\xi_N\Big)\,dy_N, \tag{3.13}$$

$$\widehat{V_N(\psi)} = -\int_{-\infty}^{\infty} \widehat{\psi}(\xi', y_N)\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi_N e^{i(x_N-y_N)\xi_N}}{|\xi|^2}\,d\xi_N\Big)\,dy_N,$$

where $\widehat{V_J(\psi)} = \widehat{V_J(\psi)}(\xi', x_N)$. Since it holds, by the residue theorem, that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ia\xi_N}}{|\xi|^2}\,d\xi_N = \frac{e^{-|a|A}}{2A}, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\xi_N e^{ia\xi_N}}{|\xi|^2}\,d\xi_N = -\mathrm{sign}(a)\frac{e^{-|a|A}}{2},$$

for $a \in \mathbf{R}\setminus\{0\}$, we insert these formulas into (3.13) in order to obtain

$$\widehat{V_j(\psi)}(\xi', x_N) = -\frac{i\xi_j}{2A}\int_{-\infty}^{\infty} e^{-|x_N-y_N|A}\widehat{\psi}(\xi', y_N)\,dy_N,$$

$$\widehat{V_N(\psi)}(\xi', x_N) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}(x_N-y_N)e^{-|x_N-y_N|A}\widehat{\psi}(\xi', y_N)\,dy_N.$$

This, combined with (3.12), furnishes that

$$V_j(\psi) = -\mathcal{F}^{-1}_{\xi'}\Big[\frac{i\xi_j}{4A}[\![\widehat{h}]\!](\xi', 0)\int_0^{\infty}\Big(e^{-|x_N-y_N|A} - e^{-|x_N+y_N|A}\Big)e^{-By_N}\,dy_N\Big],$$

$$\begin{aligned} V_N(\psi) = \mathcal{F}^{-1}_{\xi'}\Big[\frac{[\![\widehat{h}]\!](\xi', 0)}{4}\int_0^{\infty}\Big(&\mathrm{sign}(x_N-y_N)e^{-|x_N-y_N|A} \\ &- \mathrm{sign}(x_N+y_N)e^{-|x_N+y_N|A}\Big)e^{-By_N}\,dy_N\Big](x'). \end{aligned}$$

By direct calculations, we have

Lemma 3.6. *Let* $0 < \varepsilon < \pi/2$ *and* $\xi' \in \mathbf{R}^{N-1}\setminus\{0\}$. *We set*

$$A = |\xi'|, \quad B = \sqrt{\lambda + |\xi'|^2}, \quad \mathcal{M}(a) = \frac{e^{-Ba} - e^{-Aa}}{B-A} \quad (\lambda \in \Sigma_\varepsilon,\ a > 0).$$

Then it holds that, for $\pm x_N > 0$,

$$\int_0^{\infty}\Big(e^{-|x_N-y_N|A} - e^{-|x_N+y_N|A}\Big)e^{-By_N}\,dy_N = \mp\frac{2A}{B+A}\mathcal{M}(\pm x_N),$$

$$\begin{aligned} &\int_0^{\infty}\Big(\mathrm{sign}(x_N-y_N)e^{-|x_N-y_N|A} - \mathrm{sign}(x_N+y_N)e^{-|x_N+y_N|A}\Big)e^{-By_N}\,dy_N \\ &= -\frac{2A}{B+A}\mathcal{M}(\pm x_N) - \frac{2}{B+A}e^{\mp Bx_N}. \end{aligned}$$

This lemma yields that, for $\pm x_N > 0$ and $j = 1, \dots, N-1$,

$$[V_j(\psi)](x', x_N) = \pm \mathcal{F}^{-1}_{\xi'}\Big[\Big(\frac{i\xi_j}{2A(B+A)}\Big) A\mathcal{M}(\pm x_N)[\![\widehat{h}]\!](\xi', 0)\Big](x'),$$

$$[V_N(\psi)](x', x_N) = -\frac{1}{2}\mathcal{F}^{-1}_{\xi'}\Big[\Big(\frac{1}{B+A}\Big) A\mathcal{M}(\pm x_N)[\![\widehat{h}]\!](\xi', 0)\Big](x')$$

$$-\frac{1}{2}\mathcal{F}^{-1}_{\xi'}\Big[\frac{e^{\mp Bx_N}}{B}[\![\widehat{h}]\!](\xi', 0)\Big](x') + \frac{1}{2}\mathcal{F}^{-1}_{\xi'}\Big[\frac{1}{B(B+A)} A e^{\mp Bx_N}[\![\widehat{h}]\!](\xi', 0)\Big](x'),$$

where we have used the fact that $1/(B+A) = (1/B) - A/\{B(B+A)\}$. By [28, Lemmas 4.6, 4.8, 5.1, 5.2, 5.3], there exist

$$\mathbf{V}_J(\lambda) \in \mathrm{Hol}\,(\Sigma_\varepsilon, \mathcal{L}(L_q(\dot{\mathbf{R}}^N)^{N^2+N}, W^2_q(\dot{\mathbf{R}}^N))) \quad (J = 1, \dots, N)$$

such that $V_J(\psi) = \mathbf{V}_J(\lambda)(\nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})$ and that, for $l = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(L_q(\dot{\mathbf{R}}^N)^{N^2+N}, L_q(\dot{\mathbf{R}}^N)^{N^2+N+1})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l\Big(R_\lambda \mathbf{V}_J(\lambda)\Big) : \lambda \in \Sigma_\varepsilon\Big\}\Big) \le \gamma_1.$$

Recalling Remark 2.3 (1), we set, for $(H_2, H_3) \in L_q(\dot{\mathbf{R}}^N)^{N^2} \times L_q(\dot{\mathbf{R}}^N)^N$,

$$\mathbf{V}^2(\lambda)(H_2, H_3) = \big(\mathbf{V}_1(\lambda)(H_2, H_3), \dots, \mathbf{V}_N(\lambda)(H_2, H_3)\big)^T.$$

Then, $\mathbf{V}(\lambda)\mathbf{H} = \mathbf{V}^1(\lambda)H_1 + \mathbf{V}^2(\lambda)(H_2, H_3)$, $\mathbf{H} = (H_1, H_2, H_3) \in \mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$, satisfies (3.9). Moreover, for $(\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$, $V(g) = \mathbf{V}(\lambda)(\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})$ with the solution g of (3.8). □

We here set $\mathbf{S}_I(\lambda)\mathbf{H} = \mathbf{V}(\lambda)\mathbf{H} + \mathcal{T}_I(\lambda)\,(H_1, R_\lambda\mathbf{V}(\lambda)\mathbf{H}, H_2, H_3)$ for $\mathbf{H} = (H_1, H_2, H_3) \in \mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$. Then, Lemma 3.4 and Proposition 3.5, together with Proposition 2.4, shows that $\mathbf{S}_I(\lambda)$ is the required operator in Theorem 3.1. This completes the proof of Theorem 3.1. □

4. Reduced Stokes resolvent equations on a bent space

Let $\Phi : \mathbf{R}^N_x \to \mathbf{R}^N_y$ be a bijection of C^1 class and let Φ^{-1} be its inverse map, where subscripts x, y denote their variables, here and subsequently. Writing $(\nabla_x\Phi)(x) = \mathbf{A} + \mathbf{B}(x)$ and $(\nabla_y\Phi^{-1})(\Phi(x)) = \mathbf{A}_{-1} + \mathbf{B}_{-1}(x)$, we assume that $\mathbf{A}$ and $\mathbf{A}_{-1}$ are orthonormal matrices with constant coefficients and $\det\mathbf{A} = \det\mathbf{A}_{-1} = 1$, and also assume that $\mathbf{B}(x)$ and $\mathbf{B}_{-1}(x)$ are matrices of functions in $W^1_r(\mathbf{R}^N)$ with $N < r < \infty$ such that

$$\|(\mathbf{B}, \mathbf{B}_{-1})\|_{L_\infty(\mathbf{R}^N)} \le M_1, \quad \|\nabla_x(\mathbf{B}, \mathbf{B}_{-1})\|_{L_r(\mathbf{R}^N)} \le M_2. \tag{4.1}$$

We will choose M_1 small enough eventually, so that we may assume that $0 < M_1 \le 1 \le M_2$ in the following.

Remark 4.1. Since $x = \Phi^{-1}(\Phi(x))$, we have $\mathbf{I} = (\nabla_y\Phi^{-1})(\nabla_x\Phi)$. This implies that $(\nabla_y\Phi^{-1})^{-1} = (\nabla_x\Phi)$, i.e. $(\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^{-1} = \mathbf{A} + \mathbf{B}(x)$.

Set $\Omega_\pm = \Phi(\mathbf{R}^N_\pm)$ and $\Gamma = \Phi(\mathbf{R}^N_0)$, and let $\widetilde{\mathbf{n}} = \widetilde{\mathbf{n}}(y)$ be the unit normal vector on Γ, which points from Ω_+ to Ω_-. In addition, setting $\Phi^{-1} = (\Phi_{-1,1}, \dots, \Phi_{-1,N})^T$, we see that Γ is represented by $\Phi_{-1,N}(y) = 0$. Thus,

$$\widetilde{\mathbf{n}}(\Phi(x)) = -\frac{\nabla_y\Phi_{-1,N}}{|\nabla_y\Phi_{-1,N}|} = \frac{(\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^T\mathbf{n}_0}{|(\mathbf{A}_{-1} + \mathbf{B}_{-1}(x))^T\mathbf{n}_0|}, \tag{4.2}$$

with $\mathbf{n}_0 = (0, \dots, 0, -1)^T$, where we have set $\mathbf{A}_{-1} = (A_{ij})$ and $\mathbf{B}_{-1}(x) = (B_{ij}(x))$. In particular, $\widetilde{\mathbf{n}}$ is defined on $\mathbf{R}^N$ by (4.2). Since $\sum_{i=1}^N(A_{Ni} + B_{Ni}(x))^2 = 1 + \sum_{i=1}^N(2A_{Ni}B_{Ni}(x) + B_{Ni}(x)^2)$ by the fact that $\mathbf{A}_{-1}$ is a orthonormal matrix, we see by (4.1) and (4.2) that $\|\nabla_x\widetilde{\mathbf{n}}\|_{L_r(\mathbf{R}^N)} \le C_NM_2$. Let $\widetilde{\mu}_\pm = \widetilde{\mu}_\pm(y)$ be viscosity coefficients defined on $\mathbf{R}^N$ and satisfy

$$\begin{aligned} &\frac{1}{2}\mu_{\pm1} \le \widetilde{\mu}_\pm(y) \le \frac{3}{2}\mu_{\pm2} \quad (y \in \mathbf{R}^N), \\ &|\widetilde{\mu}_\pm(y) - \mu_{\pm0}| \le M_1 \quad (y \in \mathbf{R}^N), \quad \|\nabla_y\widetilde{\mu}_\pm\|_{L_r(\mathbf{R}^N)} \le C_r, \end{aligned} \tag{4.3}$$

where $\mu_{\pm0}$ are some constants with $\mu_{\pm1} \le \mu_{\pm0} \le \mu_{\pm2}$, respectively, for the same constants $\mu_{\pm1}$, $\mu_{\pm2}$ as in Theorem 1.6. In addition, we set

$$\widetilde{\mu}(y) = \widetilde{\mu}_+(y)\chi_{\Omega_+}(y) + \widetilde{\mu}_-(y)\chi_{\Omega_-}(y), \;\; \widetilde{\rho}(y) = \rho_+\chi_{\Omega_+}(y) + \rho_-\chi_{\Omega_-}(y) \tag{4.4}$$

for positive constants $\rho_\pm$, and also set $\mu_\pm(x) = \widetilde{\mu}_\pm(\Phi(x))$, $\mu(x) = \widetilde{\mu}(\Phi(x))$, $\rho(x) = \widetilde{\rho}(\Phi(x))$, and $\mu_0(x) = \widetilde{\mu}_0(\Phi(x))$ for $\widetilde{\mu}_0(y) = \mu_{+0}\chi_{\Omega_+}(y) + \mu_{-0}\chi_{\Omega_-}(y)$. It then holds that

$$\begin{aligned} &\rho = \rho(x) = \rho_+\chi_{\mathbf{R}^N_+}(x) + \rho_-\chi_{\mathbf{R}^N_-}(x), \\ &\mu_0 = \mu_0(x) = \mu_{+0}\chi_{\mathbf{R}^N_+}(x) + \mu_{-0}\chi_{\mathbf{R}^N_-}(x), \\ &\mu(x) = \mu_+(x)\chi_{\mathbf{R}^N_+}(x) + \mu_-(x)\chi_{\mathbf{R}^N_-}(x), \\ &|\mu(x) - \mu_0| \le M_1 \quad (x \in \dot{\mathbf{R}}^N), \quad \|\nabla_x\mu\|_{L_r(\dot{\mathbf{R}}^N)} \le C_r. \end{aligned} \tag{4.5}$$

In this section, we consider the two-phase reduced Stokes equation in $\dot\Omega = \Omega_+ \cup \Omega_-$ with an interface condition:

$$\begin{cases} \lambda\widetilde{\mathbf{u}} - \widetilde{\rho}^{-1}\,\mathrm{Div}\,\widetilde{\mathbf{T}}(\widetilde{\mathbf{u}}, \widetilde{K}_I(\widetilde{\mathbf{u}})) = \widetilde{\mathbf{f}} & \text{in } \dot\Omega, \\ [\![\widetilde{\mathbf{T}}(\widetilde{\mathbf{u}}, \widetilde{K}_I(\widetilde{\mathbf{u}}))\widetilde{\mathbf{n}}]\!] = [\![\widetilde{\mathbf{h}}]\!] & \text{on } \Gamma, \\ [\![\widetilde{\mathbf{u}}]\!] = 0 & \text{on } \Gamma. \end{cases} \tag{4.6}$$

Here, $\widetilde{\mathbf{T}}(\widetilde{\mathbf{u}}, \widetilde{K}_I(\widetilde{\mathbf{u}})) = \widetilde{\mu}\mathbf{D}(\widetilde{\mathbf{u}}) - \widetilde{K}_I(\widetilde{\mathbf{u}})\mathbf{I}$ and $\widetilde{K}_I(\widetilde{\mathbf{u}})$ is a unique solution to the following weak problem:

$$(\widetilde{\rho}^{-1}\nabla\widetilde{K}_I(\widetilde{\mathbf{u}}), \nabla\widetilde{\varphi})_{\dot{\Omega}} = (\widetilde{\rho}^{-1}\operatorname{Div}(\widetilde{\mu}\mathbf{D}(\widetilde{\mathbf{u}})) - \nabla\operatorname{div}\widetilde{\mathbf{u}}, \nabla\widetilde{\varphi})_{\dot{\Omega}}, \tag{4.7}$$

$$[\![\widetilde{K}_I(\widetilde{\mathbf{u}})]\!] = < [\![\widetilde{\mu}\mathbf{D}(\widetilde{\mathbf{u}})\widetilde{\mathbf{n}}]\!], \widetilde{\mathbf{n}} > - [\![\operatorname{div}\widetilde{\mathbf{u}}]\!] \quad \text{on } \Gamma, \tag{4.8}$$

for all $\widetilde{\varphi} \in \widehat{W}^1_{q'}(\mathbf{R}^N_y)$. We then have the following theorem.

Theorem 4.2. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $N < r < \infty$, and $\max(q, q') \le r$ with $q' = q/(q-1)$. Suppose that (4.1), (4.3), and (4.4) hold. Let $Z_{\mathcal{R},q}(G)$ and $\mathcal{Z}_{\mathcal{R},q}(G)$, with an open set G of $\mathbf{R}^N$, be defined as $Z_{\mathcal{R},q}(G) = L_q(G)^N \times W^1_q(G)^N$ and $\mathcal{Z}_{\mathcal{R},q}(G) = \{(H_1, \ldots, H_4) : H_1, H_3 \in L_q(G)^N, H_2 \in L_q(G)^{N^2}, H_4 \in W^1_q(G)^N\}$, respectively, while $\mu^* = 2^{-1}\min(\mu_{\pm1}, \mu_{\pm2})$. Then there exist $0 < M_1 < \min(1, \mu^*)$, $\lambda_0 \ge 1$, and an operator family $\widetilde{\mathbf{S}}_I(\lambda) \in \operatorname{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Z}_{\mathcal{R},q}(\dot{\Omega}), W^2_q(\dot{\Omega})^N))$ such that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $(\widetilde{\mathbf{f}}, \widetilde{\mathbf{h}}) \in Z_{\mathcal{R},q}(\dot{\Omega})$, $\widetilde{\mathbf{u}} = \widetilde{\mathbf{S}}_I(\lambda)H_{\mathcal{R},\lambda}(\widetilde{\mathbf{f}}, \widetilde{\mathbf{h}})$ is a unique solution to (4.6), and furthermore,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}_{\mathcal{R},q}(\dot{\Omega}), L_q(\dot{\Omega})^{\widetilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (R_\lambda\widetilde{\mathbf{S}}_I(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \le \gamma_2, \tag{4.9}$$

for $l = 0, 1$ with some positive constant γ_2. Here and subsequently, $\widetilde{N} = N^3 + N^2 + N$, $R_\lambda\mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$, and $H_{\mathcal{R},\lambda}(\widetilde{\mathbf{f}}, \widetilde{\mathbf{h}}) = (\widetilde{\mathbf{f}}, \nabla\widetilde{\mathbf{h}}, \lambda^{1/2}\widetilde{\mathbf{h}}, \widetilde{\mathbf{h}})$; M_1 is a constant depending on N, q, r, ε, $\rho_\pm$, $\mu_{\pm1}$, $\mu_{\pm2}$; λ_0 is a constant depending on M_2, N, q, r, ε, $\rho_\pm$, $\mu_{\pm1}$, $\mu_{\pm2}$; γ_2 is a generic constant depending on M_2, λ_0, N, q, r, ε, $\rho_\pm$, $\mu_{\pm1}$, $\mu_{\pm2}$.

The remaining part of this section is mainly devoted to the proof of Theorem 4.2. We rewrite the problem (4.6) as follows:

$$\begin{cases} \lambda\widetilde{\mathbf{u}} - \widetilde{\rho}^{-1}\widetilde{\mu}\operatorname{Div}\mathbf{D}(\widetilde{\mathbf{u}}) + \widetilde{\rho}^{-1}\nabla\widetilde{\theta} - \widetilde{\rho}^{-1}\mathbf{D}(\widetilde{\mathbf{u}})\nabla\widetilde{\mu} = \widetilde{\mathbf{f}} & \text{in } \dot{\Omega}, \\ [\![(\widetilde{\mu}\mathbf{D}(\widetilde{\mathbf{u}}) - \widetilde{\theta}\mathbf{I})\widetilde{\mathbf{n}}]\!] = [\![\widetilde{\mathbf{h}}]\!] & \text{on } \Gamma, \\ [\![\widetilde{\mathbf{u}}]\!] = 0 & \text{on } \Gamma, \end{cases} \tag{4.10}$$

with $\widetilde{\theta} = \widetilde{K}_I(\widetilde{\mathbf{u}})$. By the change of variable: $y = \Phi(x)$, we transform the problem (4.10) to some problem on $\dot{\mathbf{R}}^N$ for $\mathbf{u}(x) = \widetilde{\mathbf{u}}(y)$ and $\theta(x) = \widetilde{\theta}(y)$.

By direct calculations, we see that $\mathbf{v} = \mathbf{A}_{-1}\mathbf{u}$ and θ satisfy

$$\begin{cases} \lambda\mathbf{v} - \dfrac{1}{\rho}\operatorname{Div}\mathbf{T}(\mathbf{v},\theta) - \dfrac{\mu-\mu_0}{\rho}\operatorname{Div}\mathbf{D}(\mathbf{v}) + \dfrac{1}{\rho}\mathcal{F}^1(\mathbf{v}) + \dfrac{1}{\rho}\mathcal{P}^1\nabla\theta = \mathbf{f} & \text{in } \dot{\mathbf{R}}^N, \\ [\![\mathbf{T}(\mathbf{v},\theta)\mathbf{n}_0]\!] + [\![(\mu-\mu_0)\mathbf{D}(\mathbf{v})\mathbf{n}_0]\!] + [\![\mathcal{F}^2(\mathbf{v})\mathbf{n}_0]\!] = [\![\mathbf{h}]\!] & \text{on } \mathbf{R}^N_0, \\ [\![\mathbf{v}]\!] = 0 & \text{on } \mathbf{R}^N_0, \end{cases} \tag{4.11}$$

with $\mathbf{T}(\mathbf{v},\theta) = \mu_0\mathbf{D}(\mathbf{v}) - \theta\mathbf{I}$, and also

$$\begin{aligned} &(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot{\mathbf{R}}^N} + (\rho^{-1}\mathcal{P}^2\nabla\theta, \nabla\varphi)_{\dot{\mathbf{R}}^N} \qquad (4.12)\\ &= \Big(\rho^{-1}\operatorname{Div}(\mu_0\mathbf{D}(\mathbf{v})) - \nabla\operatorname{div}\mathbf{v} + \rho^{-1}(\mu-\mu_0)\operatorname{Div}\mathbf{D}(\mathbf{v}) \\ &\quad - \rho^{-1}\mathcal{F}^1(\mathbf{v}) + \mathcal{F}^3(\mathbf{v}) + \mathcal{F}^4(\mathbf{v}), \nabla\varphi\Big)_{\dot{\mathbf{R}}^N} \quad \text{for any } \varphi \in \widehat{W}^1_{q'}(\mathbf{R}^N), \\ &[\![\theta]\!] =< [\![\mu_0\mathbf{D}(\mathbf{v})\mathbf{n}_0]\!], \mathbf{n}_0 > -[\![\operatorname{div}\mathbf{v}]\!] \qquad (4.13)\\ &\quad + < [\![(\mu-\mu_0)\mathbf{D}(\mathbf{v})\mathbf{n}_0]\!], \mathbf{n}_0 > + [\![\mathcal{F}^5(\mathbf{v})]\!] \quad \text{on } \mathbf{R}^N_0. \end{aligned}$$

Here, we have set $\mathbf{f} = \mathbf{A}_{-1}\widetilde{\mathbf{f}}\circ\Phi$, $\mathbf{h} = |(\mathbf{A}_- + \mathbf{B}_-(x))^T\mathbf{n}_0|(\mathbf{A}_- + \mathbf{B}_-(x))^{-T}\widetilde{\mathbf{h}}\circ\Phi$,

$$\begin{aligned} &\mathcal{F}^1(\mathbf{v}) = \mu(\mathcal{R}^1\nabla^2\mathbf{v} + \mathcal{S}^1\nabla\mathbf{v}) + (\mathcal{T}^1\nabla\mathbf{v})\nabla\mu, \\ &\mathcal{F}^2(\mathbf{v}) = \mu\mathcal{R}^2\nabla\mathbf{v}, \quad \mathcal{F}^3(\mathbf{v}) = \mathcal{R}^3\nabla^2\mathbf{v} + \mathcal{S}^2\nabla\mathbf{v}, \\ &\mathcal{F}^4(\mathbf{v}) = \mathcal{R}^4\big(\rho^{-1}\mu\operatorname{Div}\mathbf{D}(\mathbf{v}) - \nabla\operatorname{div}\mathbf{v} - \rho^{-1}\mathcal{F}^1(\mathbf{v}) + \mathcal{F}^3(\mathbf{v})\big), \\ &\mathcal{F}^5(\mathbf{v}) = \mu\mathcal{R}^5\nabla\mathbf{v} + \mathcal{R}^6\nabla\mathbf{v}, \end{aligned}$$

and the coefficients $\mathcal{P}^i$, $\mathcal{R}^j$, $\mathcal{S}^i$, $\mathcal{T}^1$ $(i = 1, 2, j = 1, \ldots, 6)$ satisfy

$$\begin{aligned} &\|(\mathcal{P}^i, \mathcal{R}^j)\|_{L_\infty(\mathbf{R}^N)} \leq C_N M_1, \quad \|(\nabla\mathcal{P}^i, \nabla\mathcal{R}^j, \mathcal{S}^i)\|_{L_r(\mathbf{R}^N)} \leq C_N M_2, \qquad (4.14)\\ &\|(\mathcal{T}^1, \nabla\mathcal{T}^1)\|_{L_\infty(\mathbf{R}^N)\times L_r(\mathbf{R}^N)} \leq C_N M_2. \end{aligned}$$

From now on, we solve (4.11), (4.12), and (4.13). Let $\theta_1 = K_I(\mathbf{v})$ given by the solution to (3.2)-(3.3) with $\mu = \mu_0$. Setting $\theta = K_I(\mathbf{v}) + \theta_2(\mathbf{v})$ in (4.12)-(4.13), we have the weak problem for $\theta_2 = \theta_2(\mathbf{v})$ as follows:

$$\begin{aligned} &(\rho^{-1}\nabla\theta_2, \nabla\varphi)_{\dot{\mathbf{R}}^N} + (\rho^{-1}\mathcal{P}^2\nabla\theta_2, \nabla\varphi)_{\dot{\mathbf{R}}^N} = \big(\rho^{-1}(\mu-\mu_0)\operatorname{Div}\mathbf{D}(\mathbf{v}) \\ &\quad - \rho^{-1}\mathcal{F}^1(\mathbf{v}) + \mathcal{F}^3(\mathbf{v}) + \mathcal{F}^4(\mathbf{v}) - \rho^{-1}\mathcal{P}^2\nabla K_I(\mathbf{v}), \nabla\varphi\big)_{\dot{\mathbf{R}}^N} \\ &[\![\theta_2]\!] =< [\![(\mu-\mu_0)\mathbf{D}(\mathbf{v})\mathbf{n}_0]\!], \mathbf{n}_0 > + [\![\mathcal{F}^5(\mathbf{v})]\!] \quad \text{on } \mathbf{R}^N_0, \end{aligned}$$

for any $\varphi \in \widehat{W}^1_{q'}(\mathbf{R}^N)$. Substituting $\theta = K_I(\mathbf{v}) + \theta_2(\mathbf{v})$ in (4.11), we have

$$\begin{cases} \lambda\mathbf{v} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{v}, K_I(\mathbf{v})) + \mathcal{U}^1(\mathbf{v}) = \mathbf{f} & \text{in } \dot{\mathbf{R}}^N, \\ [\![\mathbf{T}(\mathbf{v}, K_I(\mathbf{v}))\mathbf{n}_0]\!] + [\![\mathcal{U}^2(\mathbf{v})\mathbf{n}_0]\!] = [\![\mathbf{h}]\!] & \text{on } \mathbf{R}^N_0, \\ [\![\mathbf{v}]\!] = 0 & \text{on } \mathbf{R}^N_0, \end{cases} \tag{4.15}$$

where we have set

$$\begin{aligned} \mathcal{U}^1(\mathbf{v}) &= -\rho^{-1}(\mu - \mu_0)\operatorname{Div}\mathbf{D}(\mathbf{v}) + \rho^{-1}\mathcal{F}^1(\mathbf{v}) \\ &\quad + \rho^{-1}\mathcal{P}^1\nabla K_I(\mathbf{v}) + \rho^{-1}(\mathbf{I} + \mathcal{P}^1)\nabla\theta_2(\mathbf{v}), \\ \mathcal{U}^2(\mathbf{v}) &= \mathcal{F}^2(\mathbf{v}) - \{< (\mu - \mu_0)\mathbf{D}(\mathbf{v})\mathbf{n}_0, \mathbf{n}_0 > + \mathcal{F}^5(\mathbf{v})\}\mathbf{I}. \end{aligned}$$

At this point, we introduce a result about the unique solvablity of the weak problem: for all $\varphi \in \widehat{W}^1_{q'}(\mathbf{R}^N)$,

$$(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot{\mathbf{R}}^N} + (\rho^{-1}\mathcal{P}^2\nabla\theta, \nabla\varphi)_{\dot{\mathbf{R}}^N} = (\mathbf{f}, \nabla\varphi)_{\dot{\mathbf{R}}^N}, \tag{4.16}$$

$$[\![\theta]\!] = [\![g]\!] \quad \text{on } \mathbf{R}^N_0. \tag{4.17}$$

Lemma 4.3. *Let $1 < q < \infty$. Then there exist a constant $M_1 \in (0, 1)$ and an operator $\Psi \in \mathcal{L}(L_q(\dot{\mathbf{R}}^N)^N \times W^1_q(\dot{\mathbf{R}}^N), W^1_q(\dot{\mathbf{R}}^N) + \widehat{W}^1_q(\mathbf{R}^N))$ such that, for any $\mathbf{f} \in L_q(\dot{\mathbf{R}}^N)^N$ and $g \in W^1_q(\dot{\mathbf{R}}^N)$, $\theta = \Psi(\mathbf{f}, g)$ is a unique solution to* (4.16)-(4.17)*, which possesses the estimate: $\|\nabla\theta\|_{L_q(\dot{\mathbf{R}}^N)} \leq C_{N,q}(\|\mathbf{f}\|_{L_q(\dot{\mathbf{R}}^N)} + \|g\|_{W^1_q(\dot{\mathbf{R}}^N)})$ with a positive constant $C_{N,q}$ independent of M_2.*

Proof. Since the weak problem (3.2)-(3.3) is uniquely solvable, we can prove Lemma 4.3 by the small perturbation method, so that we may omit the detailed proof. □

By Lemma 4.3, we have $\theta_2(\mathbf{v}) = \Psi(\mathbf{f}, g)$ with

$$\begin{aligned} \mathbf{f} &= \rho^{-1}(\mu - \mu_0)\operatorname{Div}\mathbf{D}(\mathbf{v}) - \rho^{-1}\mathcal{F}^1(\mathbf{v}) + \mathcal{F}^3(\mathbf{v}) + \mathcal{F}^4(\mathbf{v}) - \rho^{-1}\mathcal{P}^2\nabla K_I(\mathbf{v}), \\ g &= < (\mu - \mu_0)\mathbf{D}(\mathbf{v})\mathbf{n}_0, \mathbf{n}_0 > + \mathcal{F}^5(\mathbf{v}). \end{aligned}$$

Next, we solve the problem (4.15) by using Theorem 3.1. Substituting $\mathbf{v} = \mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})$ in (4.15) yields that

$$\begin{cases} \lambda\mathbf{v} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{v}, K_I(\mathbf{v})) = \mathbf{f} - \mathcal{U}^1(\mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})) & \text{in } \dot{\mathbf{R}}^N, \\ [\![\mathbf{T}(\mathbf{v}, K_I(\mathbf{v}))\mathbf{n}_0]\!] = [\![\mathbf{h} - \mathcal{U}^2(\mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}))\mathbf{n}_0]\!] & \text{on } \mathbf{R}^N_0, \\ [\![\mathbf{v}]\!] = 0 & \text{on } \mathbf{R}^N_0. \end{cases} \tag{4.18}$$

Let $\mathcal{V}^i(\lambda)(\mathbf{f},\mathbf{h}) = \mathcal{U}^i(\mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h}))$ for $i = 1,2$, and set $\mathcal{V}(\lambda)(\mathbf{f},\mathbf{h}) = (\mathcal{V}^1(\lambda)(\mathbf{f},\mathbf{h}), \mathcal{V}^2(\lambda)(\mathbf{f},\mathbf{h}))$ and

$$Y^\lambda_{\mathcal{R},q}(\dot{\mathbf{R}}^N) = \{(\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}) : (\mathbf{f},\mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)\} \quad (\lambda \neq 0).$$

Then, for each $\lambda \neq 0$, $\varphi_\lambda(\mathbf{f},\mathbf{h}) := G_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h})$ is a bijection from $Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$ onto $Y^\lambda_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$. If there is the inverse operator of $(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1})$, then $\mathbf{v} = \mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}\varphi_\lambda^{-1}(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1})^{-1}\varphi_\lambda(\mathbf{f},\mathbf{h})$ solves (4.15) since $\varphi_\lambda^{-1}(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1})^{-1}\varphi_\lambda = (I - \mathcal{V}(\lambda))^{-1}$. In what follows, we show the invertibility above and the $\mathcal{R}$-boundedness of the inverse operator. To this end, we estimate the remainder terms on the right-hand sides of (4.18). We combine Proposition 2.6 for $\dot\Omega = \dot{\mathbf{R}}^N$ with (4.14) in order to obtain

$$\begin{aligned}
&\|\mathcal{F}^i(\mathbf{v})\|_{L_q(\dot{\mathbf{R}}^N)} \leq \gamma_3(M_1+\sigma)\|\nabla^2\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)} + \gamma_{\sigma,M_2}\|\nabla\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)}, \qquad (4.19)\\
&\|\mathcal{F}^j(\mathbf{v})\|_{L_q(\dot{\mathbf{R}}^N)} \leq \gamma_3 M_1\|\nabla\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)},\\
&\|\nabla\mathcal{F}^j(\mathbf{v})\|_{L_q(\dot{\mathbf{R}}^N)} \leq \gamma_3(M_1+\sigma)\|\nabla^2\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)} + \gamma_{\sigma,M_2}\|\nabla\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)},\\
&\|\mathcal{P}^k\nabla K_I(\mathbf{v})\|_{L_q(\dot{\mathbf{R}}^N)} \leq \gamma_3 M_1\|\nabla\mathbf{v}\|_{W^1_q(\dot{\mathbf{R}}^N)},
\end{aligned}$$

for $i = 1,3,4$, $j = 2,5$, $k = 1,2$. Here, and subsequently, γ_3 is a generic constant depending, at most, on N, q, r, ρ_+, ρ_-, μ_{+1}, μ_{+2}, μ_{+2}, and μ_{-2}; γ_{σ,M_2} is a generic constant depending, at most, on M_2, σ, N, q, r, ρ_+, ρ_-, μ_{+1}, μ_{+2}, μ_{+2}, and μ_{-2}. In addition, by Lemma 4.3, (4.19), and (4.5), together with Proposition 2.6,

$$\|(I+\mathcal{P}^1)\nabla\theta_2(\mathbf{v})\|_{L_q(\dot{\mathbf{R}}^N)} \leq \gamma_3(M_1+\sigma)\|\nabla^2\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)} + \gamma_{\sigma,M_2}\|\nabla\mathbf{v}\|_{L_q(\dot{\mathbf{R}}^N)}. \qquad (4.20)$$

We define operators $\mathbf{V}^i(\lambda)$, $i = 1,2$, as $\mathbf{V}^i(\lambda)\mathbf{H} = \mathcal{U}^i(\mathbf{S}_I(\lambda)\mathbf{H})$ for $\mathbf{H} = (H_1,H_2,H_3) \in \mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$. Then, we have $\mathcal{V}^i(\mathbf{f},\mathbf{h}) = \mathbf{V}^i(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h})$ and have, by Proposition 2.5, (4.19), (4.20), and Theorem 3.1,

$$\begin{aligned}
&\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), L_q(\dot{\mathbf{R}}^N)^N)}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^l \mathbf{V}^1(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\right\}\right)\\
&\quad \leq \gamma_1\left(\gamma_3(M_1+\sigma) + \gamma_{\sigma,M_2}\lambda_0^{-1/2}\right),\\
&\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), L_q(\dot{\mathbf{R}}^N)^{N^2+N})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^l \left(\nabla\mathbf{V}^2(\lambda), \lambda^{1/2}\mathbf{V}^2(\lambda)\right) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\right\}\right)\\
&\quad \leq \gamma_1\left(\gamma_3(M_1+\sigma) + \gamma_{\sigma,M_2}\lambda_0^{-1/2}\right),
\end{aligned}$$

for $l = 0, 1$ and for any $\lambda_0 > 0$ (cf. [26, pp. 345-346] for more details). Thus, setting $\mathbf{V}(\lambda)\mathbf{H} = (\mathbf{V}^1(\lambda)\mathbf{H}, \mathbf{V}^2(\lambda)\mathbf{H})$ for $\mathbf{H} \in \mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$ furnishes that

$$\mathcal{V}(\lambda)(\mathbf{f}, \mathbf{h}) = \mathbf{V}(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N) \text{ for } (\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N), \tag{4.21}$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (G_{\mathcal{R},\lambda}\mathbf{V}(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big)$$
$$\leq \gamma_1\big(\gamma_3(M_1 + \sigma) + \gamma_{\sigma,M_2}\lambda_0^{-1/2}\big) \quad (l = 0, 1).$$

If we choose σ and M_1 so small that $\gamma_1\gamma_3\sigma \leq 1/8$ and $\gamma_1\gamma_3 M_1 \leq 1/8$ and if we choose $\lambda_0 \geq 1$ so large that $\gamma_{\sigma,M_2}\lambda_0^{-1/2} \leq 1/4$, then we have, by (4.21),

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (G_{\mathcal{R},\lambda}\mathbf{V}(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \leq \frac{1}{2} \quad (l = 0, 1). \tag{4.22}$$

Since it holds, by (4.21) and (4.22), that

$$\|\varphi_\lambda \mathcal{V}(\lambda)\varphi_\lambda^{-1}(\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})\|_{\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)} = \|G_{\mathcal{R},\lambda}\mathcal{V}(\lambda)(\mathbf{f}, \mathbf{h})\|_{\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)}$$
$$= \|G_{\mathcal{R},\lambda}\mathbf{V}(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})\|_{\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)} \leq \frac{1}{2}\|(\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})\|_{\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N)},$$

there exists the inverse mapping $(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1})^{-1} \in \mathcal{L}(Y^\lambda_{\mathcal{R},q}(\dot{\mathbf{R}}^N))$ for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$. In addition, $(I - G_{\mathcal{R},\lambda}(\lambda)\mathbf{V}(\lambda))^{-1} = \sum_{j=0}^\infty (G_{\mathcal{R},\lambda}\mathbf{V}(\lambda))^j$ exists by (4.22) and satisfies the estimate: for $l = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (I - G_{\mathcal{R},\lambda}\mathbf{V}(\lambda))^{-1} : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \leq 2. \tag{4.23}$$

If we set $\mathbf{v} = \mathbf{S}_I(\lambda)G_{\mathcal{R},\lambda}\varphi_\lambda^{-1}(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1})^{-1}\varphi_\lambda(\mathbf{f}, \mathbf{h})$, then $\mathbf{v}$ is a solution to (4.15) as mentioned above. Noting that $\varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1}G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}) = G_{\mathcal{R},\lambda}\mathbf{V}(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})$ by (4.21), we see that

$$G_{\mathcal{R},\lambda}\varphi_\lambda^{-1}(I - \varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1})^{-1}\varphi_\lambda(\mathbf{f}, \mathbf{h})$$
$$= \sum_{j=0}^\infty \big(\varphi_\lambda\mathcal{V}(\lambda)\varphi_\lambda^{-1}\big)^j G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}) = (I - G_{\mathcal{R},\lambda}\mathbf{V}(\lambda))^{-1}G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}).$$

Set $\mathcal{S}_I(\lambda) = \mathbf{S}_I(\lambda)(I - G_{\mathcal{R},\lambda}\mathbf{V}(\lambda))^{-1}$, and then $\mathbf{v} = \mathcal{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h})$ is a solution to (4.15) for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $(\mathbf{f}, \mathbf{h}) \in Y_{\mathcal{R},q}(\dot{\mathbf{R}}^N)$. Furthermore, by (4.23) and Theorem 3.1, we have

$$\mathcal{S}_I(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), W_q^2(\dot{\mathbf{R}}^N)^N)), \tag{4.24}$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_{\mathcal{R},q}(\dot{\mathbf{R}}^N), L_q(\dot{\mathbf{R}}^N)^{\tilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (R_\lambda\mathcal{S}_I(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \leq \gamma_2 \quad (l = 0, 1).$$

The uniqueness of solutions to (4.15) can be proved in the same manner as in [26, Section 4].

Setting $\widetilde{\mathbf{u}} = \mathbf{A}_{-1}^T\mathbf{v}\circ\Phi^{-1} = [\mathbf{A}_{-1}^T\mathcal{S}_I(\lambda)G_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h})]\circ\Phi^{-1}$ and noting $\mathbf{A}_{-1}^T = (\mathbf{A}_{-1})^{-1}$, we see that $\widetilde{\mathbf{u}}$ is a unique solution to (4.6). Recall that $\mathbf{f} = \mathbf{A}_{-1}\widetilde{\mathbf{f}}\circ\Phi$ and $\mathbf{h} = |(\mathbf{A}_{-1}+\mathbf{B}_{-1}(x))^T\mathbf{n}_0|(\mathbf{A}_{-1}+\mathbf{B}_{-1}(x))^{-T}\widetilde{\mathbf{h}}\circ\Phi$, and set $\mathbf{E}(x) = |(\mathbf{A}_{-1}+\mathbf{B}_{-1}(x))^T\mathbf{n}_0|(\mathbf{A}+\mathbf{B}(x))^T$ in view of Remark 4.1. Observing that

$$G_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h}) = \Big(\mathbf{A}_{-1}\widetilde{\mathbf{f}}\circ\Phi, (\nabla\mathbf{E}(x))\widetilde{\mathbf{h}}\circ\Phi+\mathbf{E}(x)[(\nabla\widetilde{\mathbf{h}})\circ\Phi]\nabla\Phi, \mathbf{E}(x)(\lambda^{1/2}\widetilde{\mathbf{h}})\circ\Phi\Big),$$

we define, for $\mathbf{H} = (H_1, H_2, H_3, H_4) \in \mathcal{Z}_{\mathcal{R},q}(\dot\Omega)$, an operator $\widetilde{\mathbf{S}}_I(\lambda)$ by

$$\begin{aligned}\widetilde{\mathbf{S}}_I(\lambda)\mathbf{H} = \Big[\mathbf{A}_{-1}^T\mathcal{S}_I(\lambda)\Big(&\mathbf{A}_{-1}H_1\circ\Phi,\\ &(\nabla\mathbf{E}(x))H_4\circ\Phi+\mathbf{E}(x)(H_2\circ\Phi)\nabla\Phi, \mathbf{E}(x)H_3\circ\Phi\Big)\Big]\circ\Phi^{-1}.\end{aligned}$$

Then, $\widetilde{\mathbf{S}}_I(\lambda)$ satisfies (4.9) by (4.24) and Proposition 2.6 with $\sigma = 1$, and also $\widetilde{\mathbf{u}} = \widetilde{\mathbf{S}}_I(\lambda)H_{\mathcal{R},\lambda}(\widetilde{\mathbf{f}},\widetilde{\mathbf{h}})$ solves (4.6) uniquely. This completes the proof of Theorem 4.2.

5. A PROOF OF THEOREM 2.2

As was discussed in Subsection 2.3, our main result Theorem 1.6 follows from Theorem 2.2, so that we prove Theorem 2.2 in this section.

5.1. **Some preparations for the proof of Theorem 2.2.** First, we state several properties of uniform $W_r^{2-1/r}$ domain (cf. [14, Proposition 6.1], [17]).

Proposition 5.1. *Let $N < r < \infty$ and let $\Omega_\pm$ be uniform $W_r^{2-1/r}$ domains in $\mathbf{R}^N$. Let M_1 the number given in Section 4. Then there exist constants $M_2 > 0$, $0 < d^i < 1$ $(i = 1, \ldots, 5)$, at most countably many N-vectors of functions $\Phi_j^i \in W_r^2(\mathbf{R}^N)^N$ $(j \in \mathbf{N},\ i = 1,2,3)$, $x_j^1 \in \Gamma$, $x_j^2 \in \Gamma_+$, $x_j^3 \in \Gamma_-$, $x_j^4 \in \Omega_+$, and $x_j^5 \in \Omega_-$ such that the following assertions hold:*

(1) *The maps: $\mathbf{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbf{R}^N$ $(j \in \mathbf{N},\ i = 1,2,3)$ are bijective such that $\nabla\Phi_j^i = \mathbf{A}_j^i + \mathbf{B}_j^i(x)$ and $\nabla(\Phi_j^i)^{-1} = \mathbf{A}_{j,-1}^i + \mathbf{B}_{j,-1}^i(x)$, where $\mathbf{A}_j^i$, $\mathbf{A}_{j,-1}^i$ are $N\times N$ constant orthonormal matrices and $\mathbf{B}_j^i(x)$, $\mathbf{B}_{j,-1}^i(x)$ are $N \times N$ matrices of $W_r^1(\mathbf{R}^N)$ functions which satisfy the conditions: $\|(\mathbf{B}_j^i, \mathbf{B}_{j,-1}^i)\|_{L_\infty(\mathbf{R}^N)} \le M_1$ and $\|\nabla(\mathbf{B}_j^i, \mathbf{B}_{j,-1}^i)\|_{L_r(\mathbf{R}^N)} \le M_2$.*
(2) *$\Omega = \big\{\bigcup_{i=1,2,3}\bigcup_{j=1}^\infty(\Phi_j^i(H^i)\cap B_{d^i}(x_j^i))\big\} \cup \big\{\bigcup_{i=4,5}\bigcup_{j=1}^\infty B_{d^i}(x_j^i)\big\}$ with $H^1 = \mathbf{R}^N$, $H^2 = \mathbf{R}^N_+$, and $H^3 = \mathbf{R}^N_-$, where $\Phi_j^i(\mathbf{R}^N_+)\cap B_{d^i}(x_j^i) = \Omega_+\cap B_{d^i}(x_j^i)$ $(i = 1,2)$, $\Phi_j^i(\mathbf{R}^N_-)\cap B_{d^i}(x_j^i) = \Omega_-\cap B_{d^i}(x_j^i)$ $(i = 1,3)$,*

$B_{d^4}(x_j^4) \subset \Omega_+$, $B_{d^5}(x_j^5) \subset \Omega_-$, *and* $\Phi_j^i(\mathbf{R}_0^N) \cap B_{d^i}(x_j^i) = \Gamma^i \cap B_{d^i}(x_j^i)$ $(i = 1, 2, 3)$. *Here and subsequently, we set* $\Gamma^1 = \Gamma$, $\Gamma^2 = \Gamma_+$, *and* $\Gamma^3 = \Gamma_-$ *for the notational convenience.*

(3) *There exist* C^∞ *functions* ζ_j^i *and* $\widetilde{\zeta}_j^i$ $(i = 1, \ldots, 5,\ j \in \mathbf{N})$ *such that* $\|(\zeta_j^i, \widetilde{\zeta}_j^i)\|_{W^2_\infty(\mathbf{R}^N)} \leq c_0$, $0 \leq \zeta_j^i, \widetilde{\zeta}_j^i \leq 1$, $\mathrm{supp}\,\zeta_j^i$, $\mathrm{supp}\,\widetilde{\zeta}_j^i \subset B_{d^i}(x_j^i)$, $\widetilde{\zeta}_j^i = 1$ *on* $\mathrm{supp}\,\zeta_j^i$, $\sum_{i=1,\ldots,5}\sum_{j=1}^\infty \zeta_j^i = 1$ *on* $\overline{\Omega}$, *and* $\sum_{j=1}^\infty \zeta_j^i = 1$ *on* Γ^i $(i = 1, 2, 3)$. *Here,* c_0 *is a positive constant depending on* M_2, N, *and* r, *but independent of* $j \in \mathbf{N}$.

(4) *There exists a natural number* $L \geq 2$ *such that any* $L+1$ *distinct sets of* $\{B_{d^i}(x_j^i) : i = 1, \ldots, 5,\ \ j \in \mathbf{N}\}$ *have an empty intersection.*

Since $\mu_\pm(x)$ is uniformly continuous in $\mathbf{R}^N$ as was assumed in the assumption (c), choosing $d^i > 0$ smaller, if necessary, allows us to assume that $|\mu_\pm(x) - \mu_\pm(x_j^i)| \leq M_1$ for any $x \in B_{d^i}(x_j^i)$ with $i = 1, \ldots, 5$ and $j \in \mathbf{N}$. Moreover, after choosing M_2 and d^i according to M_1 in Proposition 5.1, we choose M_2 again so large that $\|\nabla\mu_\pm\|_{L_r(B_{d^i}(x_j^i))} \leq M_2$. Here, and in the following, constants denoted by C are independent of $j \in \mathbf{N}$. In view of (4.2), we may assume that unit normal vectors $\mathbf{n}_j^i$ to $\Gamma_j^i = \Phi_j^i(\mathbf{R}_0^N)$ $(i = 1, 2, 3,$ $j \in \mathbf{N})$ are defined on $\mathbf{R}^N$ together with $\|\mathbf{n}_j^i\|_{L_\infty(\mathbf{R}^N)} = 1$, and also they satisfy, by Proposition 5.1 (1), the conditions: $\|\nabla\mathbf{n}_j^i\|_{L_r(\mathbf{R}^N)} \leq CM_2$. Note that $\mathbf{n} = \mathbf{n}_j^1$ on $B_{d^1}(x_j^1) \cap \Gamma$ and points from Ω_+ to Ω_-, and besides, the unit outward normal $\mathbf{n}_\pm$ to $\Gamma_\pm$ satisfy $\mathbf{n}_+ = \mathbf{n}_j^2$ on $B_{d^2}(x_j^2) \cap \Gamma_+$ and $\mathbf{n}_- = \mathbf{n}_j^3$ on $B_{d^3}(x_j^3) \cap \Gamma_-$, respectively.

Summing up the above properties, we suppose in this section that

$$\begin{aligned} &\mu_{\pm1} \leq \mu_\pm(x_j^i) \leq \mu_{\pm2}, \quad |\mu_\pm(x) - \mu_\pm(x_j^i)| \leq M_1 \quad (x \in B_{d^i}(x_j^i)), \\ &\|\nabla\mu_\pm\|_{L_r(B_{d^i}(x_j^i))} \leq M_2. \end{aligned} \tag{5.1}$$

Let $B_j^i = B_{d^i}(x_j^i)$ with $i = 1, \ldots, 5$ and $j \in \mathbf{N}$ for short. Then, by the finite intersection property stated in Proposition 5.1 (4), we see that, for any $s \in [1, \infty)$, there is a positive constant $C_{s,L}$ such that, for any $f \in L_s(G)$ with an open set G of $\mathbf{R}^N$ and for $i = 1, \ldots, 5$,

$$\Big(\sum_{j=1}^\infty \|f\|^s_{L_s(G\cap B_j^i)}\Big)^{1/s} \leq C_{s,L}\|f\|_{L_s(G)}. \tag{5.2}$$

Next, we prepare two lemmas used to construct parametrixes. The second one follows from Lemma 5.2 below and (5.2).

Lemma 5.2. *Let X be a Banach space and X^* its dual space, while $\|\cdot\|_X$, $\|\cdot\|_{X^*}$, and $<\cdot,\cdot>$ be the norm of X, the norm of X^*, and the duality pairing between of X and X^*, respectively. Let $n \in \mathbf{N}$, $l = 1, \ldots, n$, and $\{a_l\}_{l=1}^n \subset \mathbf{C}$, and let $\{f_j^l\}_{j=1}^\infty$ be sequences in X^* and $\{g_j^l\}_{j=1}^\infty$, $\{h_j\}_{j=1}^\infty$ be sequences of positive numbers. Assume that there exist maps $\mathcal{N}_j : X \to [0, \infty)$ such that*

$$|< f_j^l, \varphi >| \leq M_3 g_j^l \mathcal{N}_j(\varphi) \quad (l = 1, \ldots, n), \quad \Big|\Big\langle \sum_{l=1}^n a_l f_j^l, \varphi\Big\rangle\Big| \leq M_3 h_j \mathcal{N}_j(\varphi),$$

for any $\varphi \in X$ with some positive constant M_3 independent of $j \in \mathbf{N}$ and $l = 1, \ldots, n$. If $\sum_{j=1}^\infty (g_j^l)^q < \infty$, $\sum_{j=1}^\infty (h_j)^q < \infty$, and $\sum_{j=1}^\infty (\mathcal{N}_j(\varphi))^{q'} \leq (M_4\|\varphi\|_X)^{q'}$ with $1 < q < \infty$ and $q' = q/(q-1)$ for a positive constant M_4, then the infinite sum $f^l = \sum_{j=1}^\infty f_j^l$ exists in the strong topology of X^* and

$$\|f^l\|_{X^*} \leq M_3 M_4 \Big(\sum_{j=1}^\infty (g_j^l)^q\Big)^{1/q}, \quad \Big\|\sum_{l=1}^n a_l f^l\Big\|_{X^*} \leq M_3 M_4 \Big(\sum_{j=1}^\infty (h_j)^q\Big)^{1/q}.$$

Proof. Let $F_m^l = \sum_{j=1}^m f_j^l$. We see that $\{F_m^l\}_{m=1}^\infty$ is a Cauchy sequence in X^*, which implies the existence of f^l. Then the estimates hold clearly. □

Lemma 5.3. *Let $1 < q < \infty$, $q' = q/(q-1)$, $i = 1, \ldots, 5$, and $m \in \mathbf{N}_0$. Let $\{f_j\}_{j=1}^\infty$ be a sequence of $W_q^m(\dot\Omega)$ and let $\{g_j^l\}_{j=1}^\infty$ be sequences of positive numbers for $l = 0, 1 \ldots, m$. Assume that $\sum_{j=1}^\infty (g_j^l)^q < \infty$ and $|(\nabla^l f_j, \varphi)_{\dot\Omega}| \leq M_5 g_j^l \|\varphi\|_{L_{q'}(\dot\Omega \cap B_j^i)}$ for any $\varphi \in L_{q'}(\dot\Omega)$ with some positive constant M_5 independent of $j \in \mathbf{N}$ and $l = 0, 1 \ldots, m$. Then, $f = \sum_{j=1}^\infty f_j$ exists in the strong topology of $W_q^m(\dot\Omega)$ and $\|\nabla^l f\|_{L_q(\dot\Omega)} \leq C_{q,L} M_5 (\sum_{j=1}^\infty (g_j^l)^q)^{1/q}$ with some positive constant $C_{q,L}$.*

5.2. **Local solutions.** In view of (5.1), we define local viscosity coefficients $\nu_{\pm j}^i(x)$ by $\nu_{\pm j}^i(x) = (\mu_\pm(x) - \mu_\pm(x_j^i))\widetilde{\zeta}_j^i(x) + \mu_\pm(x_j^i)$. Note that $M_1 \leq (1/2)\min(\mu_{+1}, \mu_{-1}, \mu_{+2}, \mu_{-2})$ as was stated in Theorem 4.2. Then, using (5.1) and setting $\mu_{\pm j}^i = \mu_\pm(x_j^i)$, we have

$$\begin{aligned} &\frac{1}{2}\mu_{\pm 1} \leq \nu_{\pm j}^i(x) \leq \frac{3}{2}\mu_{\pm 2}, \quad |\nu_{\pm j}^i(x) - \mu_{\pm j}^i| \leq M_1 \quad (x \in \mathbf{R}^N), \\ &\|\nabla \nu_{\pm j}^i\|_{L_r(\mathbf{R}^N)} \leq C_{M_2, r}, \end{aligned} \tag{5.3}$$

with $\mu_{\pm 1} \leq \mu_{\pm j}^i \leq \mu_{\pm 2}$. The condition (5.3) implies that $\nu_{\pm j}^i(x)$ satisfy (4.3).

Set $\mathcal{H}^0_j = \Phi^1_j(\mathbf{R}^N)$, $\mathcal{H}^1_j = \mathcal{H}^1_{+j} \cup \mathcal{H}^1_{-j}$ ($\mathcal{H}^1_{\pm j} = \Phi^1_j(\mathbf{R}^N_\pm)$), $\mathcal{H}^2_j = \Phi^2_j(\mathbf{R}^N_+)$, $\mathcal{H}^3_j = \Phi^3_j(\mathbf{R}^N_-)$, $\mathcal{H}^4_j = \mathcal{H}^5_j = \mathbf{R}^N$, $\Gamma^1_j = \Phi^1_j(\mathbf{R}^N_0)$, $\Gamma^2_j = \Phi^2_j(\mathbf{R}^N_0)$, and $\Gamma^3_j = \Phi^3_j(\mathbf{R}^N_0)$ in what follows. Let us define $\nu^i_j(x)$ and $\rho^i_j(x)$ as follows: $\nu^1_j(x) = \nu^1_{+j}(x)\chi_{\mathcal{H}^1_{+j}}(x) + \nu^1_{-j}(x)\chi_{\mathcal{H}^1_{-j}}(x)$, $\nu^i_j(x) = \nu^i_{+j}(x)$ $(i = 2, 4)$, $\nu^i_j(x) = \nu^i_{-j}(x)$ $(i = 3, 5)$; $\rho^1_j(x) = \rho_+\chi_{\mathcal{H}^1_{+j}}(x) + \rho_-\chi_{\mathcal{H}^1_{-j}}(x)$, $\rho^i_j(x) = \rho_+$ $(i = 2, 4)$, $\rho^i_j(x) = \rho_-$ $(i = 3, 5)$. We then see that, for $i = 1, \ldots, 5$, $j \in \mathbf{N}$, and $x \in \operatorname{supp}\zeta^i_j$,

$$\begin{aligned} \nu^i_j(x) &= \mu(x) = \mu_+(x)\chi_{\Omega_+}(x) + \mu_-(x)\chi_{\Omega_-}(x), \\ \rho^i_j(x) &= \rho(x) = \rho_+\chi_{\Omega_+}(x) + \rho_-\chi_{\Omega_-}(x), \end{aligned} \tag{5.4}$$

because $\widetilde{\zeta}^i_j = 1$ on $\operatorname{supp}\zeta^i_j$. Moreover, we set $\mathbf{T}^i_j(\mathbf{u}, \theta) = \nu^i_j(x)\mathbf{D}(\mathbf{u}) - \theta\mathbf{I}$.

Let $(\mathbf{f}, \mathbf{h}, \mathbf{k}) \in X_{\mathcal{R},q}(\dot\Omega)$. We consider the following problems:

$$\begin{cases} \lambda\mathbf{u}^1_j - (\rho^1_j)^{-1}\operatorname{Div}\mathbf{T}^1_j(\mathbf{u}^1_j, K^1_j(\mathbf{u}^1_j)) = \widetilde{\zeta}^1_j\mathbf{f} & \text{in } \mathcal{H}^1_j, \\ [\![\mathbf{T}^1_j(\mathbf{u}^1_j, K^1_j(\mathbf{u}^1_j))\mathbf{n}^1_j]\!] = \widetilde{\zeta}^1_j\mathbf{h} & \text{on } \Gamma^1_j, \\ [\![\mathbf{u}^1_j]\!] = 0 & \text{on } \Gamma^1_j, \end{cases} \tag{5.5}$$

and furthermore,

$$\lambda\mathbf{u}^2_j - (\rho^2_j)^{-1}\operatorname{Div}\mathbf{T}^2_j(\mathbf{u}^2_j, K^2_j(\mathbf{u}^2_j)) = \widetilde{\zeta}^2_j\mathbf{f} \quad \text{in } \mathcal{H}^2_j, \tag{5.6}$$
$$\mathbf{T}^2_j(\mathbf{u}^2_j, K^2_j(\mathbf{u}^2_j))\mathbf{n}^2_j = \widetilde{\zeta}^2_j\mathbf{k} \quad \text{on } \Gamma^2_j,$$
$$\lambda\mathbf{u}^3_j - (\rho^3_j)^{-1}\operatorname{Div}\mathbf{T}^3_j(\mathbf{u}^3_j, K^3_j(\mathbf{u}^3_j)) = \widetilde{\zeta}^3_j\mathbf{f} \quad \text{in } \mathcal{H}^3_j, \tag{5.7}$$
$$\mathbf{u}^3_j = 0 \quad \text{on } \Gamma^3_j,$$
$$\lambda\mathbf{u}^4_j - (\rho^4_j)^{-1}\operatorname{Div}\mathbf{T}^4_j(\mathbf{u}^4_j, K^4_j(\mathbf{u}^4_j)) = \widetilde{\zeta}^4_j\mathbf{f} \quad \text{in } \mathcal{H}^4_j, \tag{5.8}$$
$$\lambda\mathbf{u}^5_j - (\rho^5_j)^{-1}\operatorname{Div}\mathbf{T}^5_j(\mathbf{u}^5_j, K^5_j(\mathbf{u}^5_j)) = \widetilde{\zeta}^5_j\mathbf{f} \quad \text{in } \mathcal{H}^5_j. \tag{5.9}$$

Here, $K^i_j(\mathbf{u}^i_j)$ $(i = 1, \ldots, 5,\ j \in \mathbf{N})$ are given as follows: For $\mathbf{u}^1_j \in W^2_q(\mathcal{H}^1_j)^N$, $K^1_j(\mathbf{u}^1_j) \in W^1_q(\mathcal{H}^1_j) + \widehat{W}^1_q(\mathcal{H}^0_j)$ denotes the solution to the weak problem:

$$\begin{aligned} &((\rho^1_j)^{-1}\nabla K^1_j(\mathbf{u}^1_j), \nabla\varphi)_{\mathcal{H}^1_j} \\ &= ((\rho^1_j)^{-1}\operatorname{Div}(\nu^1_j\mathbf{D}(\mathbf{u}^1_j)) - \nabla\operatorname{div}\mathbf{u}^1_j, \nabla\varphi)_{\mathcal{H}^1_j} \quad \text{for all } \varphi \in \widehat{W}^1_{q'}(\mathcal{H}^0_j) \end{aligned} \tag{5.10}$$

$$[\![K^1_j(\mathbf{u}^1_j)]\!] = < [\![\nu^1_j\mathbf{D}(\mathbf{u}^1_j)\mathbf{n}^1_j]\!], \mathbf{n}^1_j > - [\![\operatorname{div}\mathbf{u}^1_j]\!] \quad \text{on } \Gamma^1_j \tag{5.11}$$

with $\|\nabla K^1_j(\mathbf{u}^1_j)\|_{L_q(\mathcal{H}^1_j)} \leq C\|\nabla \mathbf{u}^1_j\|_{W^1_q(\mathcal{H}^1_j)}$; For $\mathbf{u}^2_j \in W^2_q(\mathcal{H}^2_j)^N$, $K^2_j(\mathbf{u}^2_j) \in W^1_q(\mathcal{H}^2_j) + \widehat{W}^1_{q,0}(\mathcal{H}^2_j)$ solves the weak problem: for all $\varphi \in \widehat{W}^1_{q',0}(\mathcal{H}^2_j)$,

$$((\rho^2_j)^{-1}\nabla K^2_j(\mathbf{u}^2_j), \nabla\varphi)_{\mathcal{H}^2_j} = ((\rho^2_j)^{-1}\operatorname{Div}(\nu^2_j\mathbf{D}(\mathbf{u}^2_j)) - \nabla \operatorname{div}\mathbf{u}^2_j, \nabla\varphi)_{\mathcal{H}^2_j}$$
$$K^2_j(\mathbf{u}^2_j) = <\nu^2_j\mathbf{D}(\mathbf{u}^2_j)\mathbf{n}^2_j, \mathbf{n}^2_j> - \operatorname{div}\mathbf{u}^2_j \quad \text{on } \Gamma^2_j$$

with $\|\nabla K^2_j(\mathbf{u}^2_j)\|_{L_q(\mathcal{H}^2_j)} \leq C\|\nabla \mathbf{u}^2_j\|_{L_q(\mathcal{H}^2_j)}$; For $\mathbf{u}^i_j \in W^2_q(\mathcal{H}^i_j)^N$ $(i = 3, 4, 5)$, $K^i_j(\mathbf{u}^i_j) \in \widehat{W}^1_q(\mathcal{H}^i_j)$ is the solution to the weak problem: for all $\varphi \in \widehat{W}^1_{q'}(\mathcal{H}^i_j)$,

$$((\rho^i_j)^{-1}\nabla K^i_j(\mathbf{u}^i_j), \nabla\varphi)_{\mathcal{H}^i_j} = ((\rho^i_j)^{-1}\operatorname{Div}(\nu^i_j\mathbf{D}(\mathbf{u}^i_j)) - \nabla \operatorname{div}\mathbf{u}^i_j, \nabla\varphi)_{\mathcal{H}^i_j}$$

with $\|\nabla K^i_j(\mathbf{u}^i_j)\|_{L_q(\mathcal{H}^i_j)} \leq C\|\nabla \mathbf{u}^i_j\|_{L_q(\mathcal{H}^i_j)}$.

We know that the following properties hold for (5.5)-(5.9)[2]: There exist a positive constant $\lambda_0 \geq 1$ and $\mathbf{S}^i_j(\lambda) \in \operatorname{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{Z}^i_q(\mathcal{H}^i_j), W^2_q(\mathcal{H}^i_j)^N))$ with

$$\mathcal{Z}^i_q(\mathcal{H}^i_j) = \mathcal{Z}_{\mathcal{R},q}(\mathcal{H}^i_j) \quad (i = 1, 2), \quad \mathcal{Z}^i_q(\mathcal{H}^i_j) = L_q(\mathcal{H}^i_j)^N \quad (i = 3, 4, 5),$$

such that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$,

$$\begin{aligned} &\mathbf{u}^1_j = \mathbf{S}^1_j(\lambda)H_{\mathcal{R},\lambda}(\widetilde{\zeta}^1_j\mathbf{f}, \widetilde{\zeta}^1_j\mathbf{h}), \quad \mathbf{u}^2_j = \mathbf{S}^2_j(\lambda)H_{\mathcal{R},\lambda}(\widetilde{\zeta}^1_j\mathbf{f}, \widetilde{\zeta}^1_j\mathbf{k}), \\ &\mathbf{u}^i_j = \mathbf{S}^i_j(\lambda)(\widetilde{\zeta}^i_j\mathbf{f}) \quad (i = 3, 4, 5), \end{aligned} \tag{5.12}$$

are unique solutions to (5.5)-(5.9), respectively, where $\mathcal{Z}_{\mathcal{R},q}$ and $H_{R,\lambda}$ are given in Theorem 4.2. In addition, for $l = 0, 1$,

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}^i_q(\mathcal{H}^i_j), L_q(\mathcal{H}^i_j)^{\tilde{N}})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l\Big(R_\lambda\mathbf{S}^i_j(\lambda)\Big) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \leq \gamma_4, \tag{5.13}$$

with some positive constant γ_4 depending on λ_0, but independent of $i = 1, \ldots, 5$ and $j \in \mathbf{N}$. Since the $\mathcal{R}$-boundedness implies the usual boundedness, we have, by (5.12) and (5.13) with $l = 0$,

$$\begin{aligned} \|R_\lambda\mathbf{u}^1_j\|_{L_q(\mathcal{H}^1_j)} &\leq \gamma_4\big(\|(\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h})\|_{L_q(\dot{\Omega}\cap B^1_j)} + \|\mathbf{h}\|_{W^1_q(\dot{\Omega}\cap B^1_j)}\big), \\ \|R_\lambda\mathbf{u}^2_j\|_{L_q(\mathcal{H}^2_j)} &\leq \gamma_4\big(\|\mathbf{f}\|_{L_q(\dot{\Omega}\cap B^2_j)} + \|(\nabla\mathbf{k}, \lambda^{1/2}\mathbf{k})\|_{L_q(\Omega_+\cap B^2_j)} + \|\mathbf{k}\|_{W^1_q(\Omega_+\cap B^2_j)}\big), \\ \|R_\lambda\mathbf{u}^i_j\|_{L_q(\mathcal{H}^i_j)} &\leq \gamma_4\|\mathbf{f}\|_{L_q(\dot{\Omega}\cap B^i_j)} \quad (i = 3, 4, 5) \end{aligned} \tag{5.14}$$

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[2] The existence of $\mathcal{R}$-bounded solution operator families $\mathbf{S}^1_j(\lambda)$, $\mathbf{S}^2_j(\lambda)$, $\mathbf{S}^3_j(\lambda)$ below follows from Theorem 4.2 and [26, Theorem 4.1, Theorem 4.4], respectively. In addition, concerning $\mathbf{S}^4_j(\lambda)$, $\mathbf{S}^5_j(\lambda)$, we can construct such $\mathcal{R}$-bounded solution operator families with variable viscosities in $\mathbf{R}^N$ under the same condition as (4.3) by using Theorem 3.1 similarly to Section 4.

for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, noting $|\lambda|^{-1/2} \leq \lambda_0^{-1/2}$.

5.3. **Construction of parametrices.** For $(\mathbf{f}, \mathbf{h}, \mathbf{k}) \in X_{\mathcal{R},q}(\dot\Omega)$, we consider the two-phase reduced Stokes equations (2.3). By Lemma 5.3, together with (5.2), (5.14), we see that the infinite sum $\sum_{i=1}^5 \sum_{j=1}^\infty \zeta_j^i \mathbf{u}_j^i$ exists in the strong topology of $W_q^2(\dot\Omega)^N$, so that let us define $\mathbf{u}$ by

$$\mathbf{u} = \sum_{i=1}^5 \sum_{j=1}^\infty \zeta_j^i \mathbf{u}_j^i \quad \text{in } W_q^2(\dot\Omega)^N.$$

Then, by (5.4), $\mathbf{n} = \mathbf{n}_j^1$ on $\operatorname{supp} \zeta_j^1 \cap \Gamma$, and $\mathbf{n}_+ = \mathbf{n}_j^2$ on $\operatorname{supp} \zeta_j^2 \cap \Gamma_+$,

$$\begin{cases} \lambda \mathbf{u} - \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{u}, K(\mathbf{u})) = \mathbf{f} - \mathcal{U}^0(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) & \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}]\!] = [\![\mathbf{h}]\!] - [\![\mathcal{U}^1(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k})]\!] & \text{on } \Gamma, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}_+ = \mathbf{k} - \mathcal{U}^2(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) & \text{on } \Gamma_+, \\ \mathbf{u} = 0 & \text{on } \Gamma_-, \end{cases}$$

where we have set

$$\begin{aligned} \mathcal{U}^i(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) =& \mathcal{V}^i(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) + \mathcal{P}^i(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) \quad (i = 0, 1, 2), \qquad (5.15) \\ \mathcal{V}^0(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) =& \sum_{i=1}^5 \sum_{j=1}^\infty (\rho_j^i)^{-1} \Big\{ \zeta_j^i \operatorname{Div}(\nu_j^i \mathbf{D}(\mathbf{u}_j^i)) - \operatorname{Div}(\nu_j^i \mathbf{D}(\zeta_j^i \mathbf{u}_j^i)) \Big\}, \\ \mathcal{V}^i(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) =& \sum_{j=1}^\infty \Big\{ \nu_j^i \mathbf{D}(\zeta_j^i \mathbf{u}_j^i)\mathbf{n}_j^i - \zeta_j^i \nu_j^i \mathbf{D}(\mathbf{u}_j^i)\mathbf{n}_j^i \Big\} \quad (i = 1, 2), \\ \mathcal{P}^0(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) =& \sum_{i=1}^5 \sum_{j=1}^\infty (\rho_j^i)^{-1} \Big\{ \nabla K(\zeta_j^i \mathbf{u}_j^i) - \zeta_j^i \nabla K_j^i(\mathbf{u}_j^i) \Big\}, \\ \mathcal{P}^i(\lambda)(\mathbf{f}, \mathbf{h}, \mathbf{k}) =& \sum_{j=1}^\infty \Big\{ \zeta_j^i K_j^i(\mathbf{u}_j^i)\mathbf{n}_j^i - K(\zeta_j^i \mathbf{u}_j^i)\mathbf{n}_j^i \Big\} \quad (i = 1, 2). \end{aligned}$$

Here, we have used the fact that

$$\nabla K(\mathbf{u}) = \sum_{i=1}^5 \sum_{j=1}^\infty \nabla K(\zeta_j^i \mathbf{u}_j^i) \quad \text{in } L_q(\dot\Omega)^N,$$

$$K(\mathbf{u}) = \sum_{j=1}^\infty K(\zeta^i_j\mathbf{u}^i_j) \quad \text{in } W^{1-1/q}_q(\Gamma^i) \quad (i=1,2)$$

(cf. [26, Subsection 5.3] for more details). Now, it holds that, by (5.12),

$$\begin{aligned}\mathbf{u} &= \sum_{j=1}^\infty \zeta^1_j\mathbf{S}^1_j(\lambda)\left(\widetilde{\zeta}^1_j\mathbf{f}, \widetilde{\zeta}^1_j(\nabla\mathbf{h}) + \lambda^{-1/2}(\nabla\widetilde{\zeta}^1_j)(\lambda^{1/2}\mathbf{h}), \widetilde{\zeta}^1_j(\lambda^{1/2}\mathbf{h}), \widetilde{\zeta}^1_j\mathbf{h}\right) \\ &+ \sum_{j=1}^\infty \zeta^2_j\mathbf{S}^2_j(\lambda)\left(\widetilde{\zeta}^2_j\mathbf{f}, \widetilde{\zeta}^2_j(\nabla\mathbf{k}) + \lambda^{-1/2}(\nabla\widetilde{\zeta}^2_j)(\lambda^{1/2}\mathbf{k}), \widetilde{\zeta}^2_j(\lambda^{1/2}\mathbf{k}), \widetilde{\zeta}^2_j\mathbf{k}\right) \\ &+ \sum_{i=3}^5\sum_{j=1}^\infty \zeta^i_j\mathbf{S}^i_j(\lambda)\left(\widetilde{\zeta}^i_j\mathbf{f}\right),\end{aligned}$$

so that we set, for[3] $\mathbf{H} = (H_1, \ldots, H_7) \in \mathcal{X}_{\mathcal{R},q}(\dot\Omega)$,

$$\begin{aligned}\mathcal{S}^1_j(\lambda)\mathbf{H} &= \mathbf{S}^1_j(\lambda)\big(\widetilde{\zeta}^1_jH_1, \widetilde{\zeta}^1_jH_2 + \lambda^{-1/2}(\nabla\widetilde{\zeta}^1_j)H_3, \widetilde{\zeta}^1_jH_3, \widetilde{\zeta}^1_jH_4\big), \\ \mathcal{S}^2_j(\lambda)\mathbf{H} &= \mathbf{S}^2_j(\lambda)\big(\widetilde{\zeta}^2_jH_1, \widetilde{\zeta}^2_jH_5 + \lambda^{-1/2}(\nabla\widetilde{\zeta}^2_j)H_6, \widetilde{\zeta}^2_jH_6, \widetilde{\zeta}^2_jH_7\big), \\ \mathcal{S}^i_j(\lambda)\mathbf{H} &= \mathbf{S}^i_j(\lambda)\big(\widetilde{\zeta}^i_jH_1\big) \quad (i=3,4,5).\end{aligned} \tag{5.16}$$

It then is clear that $\mathbf{u} = \sum_{i=1}^5\sum_{j=1}^\infty \zeta^i_j\mathcal{S}^i_j(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k})$. By (5.13) with Definition 1.2, we have

$$\int_0^1 \Big\|\sum_{l=1}^n r_l(u)R_{\lambda_l}(\zeta^i_j\mathcal{S}^i_j(\lambda_l)\mathbf{H}_l)\Big\|^q_{L_q(\dot\Omega)}\,du \le \gamma_4\int_0^1\Big\|\sum_{l=1}^n r_l(u)\mathbf{H}_l\Big\|^q_{\mathcal{X}_{\mathcal{R},q}(\dot\Omega\cap B^i_j)}\,du, \tag{5.17}$$

for any $n\in\mathbf{N}$, $\{\lambda_l\}_{l=1}^n \subset \Sigma_{\varepsilon,\lambda_0}$, and $\{\mathbf{H}_l\}_{l=1}^n \subset \mathcal{X}_{\mathcal{R},q}(\dot\Omega)$. The inequality (5.17) with $n=1$, together with Lemma 5.3, yields that the infinite sum $\sum_{j=1}^\infty \zeta^i_j\mathcal{S}^i_j(\lambda)\mathbf{H}$ exists in the strong topology of $W^2_q(\dot\Omega)^N$, so that let us define $\mathcal{T}^i(\lambda)\mathbf{H} = \sum_{j=1}^\infty\zeta^i_j\mathcal{S}^i_j(\lambda)\mathbf{H}$ for $i=1,\ldots,5$. In addition, by Lemma 5.2,

$$\Big\|\sum_{l=1}^n a_lR_{\lambda_l}\mathcal{T}^i(\lambda_l)\mathbf{H}_l\Big\|^q_{L_q(\dot\Omega)} \le \gamma_4\sum_{j=1}^\infty\Big\|\sum_{l=1}^n a_lR_{\lambda_l}\Big(\zeta^i_j\mathcal{S}^i_j(\lambda_l)\mathbf{H}_l\Big)\Big\|^q_{L_q(\dot\Omega)},$$

for $i=1,\ldots,5$ and for any $n\in\mathbf{N}$, $\{a_l\}_{l=1}^n\subset\mathbf{C}$, $\{\lambda_l\}_{l=1}^n\subset\Sigma_{\varepsilon,\lambda_0}$, and $\{\mathbf{H}_l\}_{l=1}^n\subset\mathcal{X}_{\mathcal{R},q}(\dot\Omega)$. The last inequality, combined with (5.2) and (5.17),

[3] As was mentioned in Remark 2.3 (1), the symbols H_1, H_2, H_3, H_4, H_5, H_6, and H_7 are variables corresponding to $\mathbf{f}$, $\nabla\mathbf{h}$, $\lambda^{1/2}\mathbf{h}$, $\mathbf{h}$, $\nabla\mathbf{k}$, $\lambda^{1/2}\mathbf{k}$, and $\mathbf{k}$, respectively.

furnishes that, by the monotone convergence theorem,

$$\int_0^1 \Big\| \sum_{l=1}^n r_l(u) R_{\lambda_l} \mathcal{T}^i(\lambda_l)\mathbf{H}_l \Big\|_{L_q(\dot\Omega)}^q\,du \le \gamma_4 \int_0^1 \Big\| \sum_{l=1}^n r_l(u)\mathbf{H}_l\Big\|_{\mathcal{X}_{\mathcal{R},q}(\dot\Omega)}^q\,du,$$

which implies that[4] $\mathcal{T}^i(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), W_q^2(\dot\Omega)^N))$ and

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), L_q(\dot\Omega)^{\widetilde N})}\left(\left\{R_\lambda \mathcal{T}^i(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\right\}\right) \le \gamma_4,$$

for $i = 1, \dots, 5$. Analogously, we see that $\{(\lambda \frac{d}{d\lambda})(R_\lambda \mathcal{T}^i(\lambda)) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), L_q(\dot\Omega)^{\widetilde N})$. Thus, setting $\mathbf{S}(\lambda)\mathbf{H} = \sum_{i=1}^5 \mathcal{T}^i(\lambda)\mathbf{H}$ yields that, by Proposition 2.4,

$$\mathbf{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), W_q^2(\dot\Omega)^N)), \quad \mathbf{u} = \mathbf{S}(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k}), \tag{5.18}$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), L_q(\dot\Omega)^{\widetilde N})}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^l \left(R_\lambda \mathbf{S}(\lambda)\right) : \lambda \in \Sigma_{\varepsilon,\lambda_0}\right\}\right) \le \gamma_4 \quad (l = 0, 1).$$

5.4. **Estimates of the remainder terms $\mathcal{U}^i(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k})$.** In this subsection, we prove the following lemma.

Lemma 5.4. *Let λ_0 and γ_4 be the same numbers as in* (5.13)*. Let $\mathcal{U}^i(\lambda)$, $\mathcal{V}^i(\lambda)$, and $\mathcal{P}^i(\lambda)$ $(i = 0, 1, 2)$ be the operators defined in* (5.15) *and set*

$$\begin{aligned}
&\mathcal{U}(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}) = \mathcal{V}(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}) + \mathcal{P}(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}),\\
&\mathcal{V}(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}) = (\mathcal{V}^0(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}), \mathcal{V}^1(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}), \mathcal{V}^2(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k})),\\
&\mathcal{P}(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}) = (\mathcal{P}^0(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}), \mathcal{P}^1(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}), \mathcal{P}^2(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k})).
\end{aligned}$$

Then, there is $\mathbf{U}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega)))$ such that

$$\begin{aligned}
&\mathcal{U}(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k}) = \mathbf{U}(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k}) \quad \textit{for } (\mathbf{f},\mathbf{h},\mathbf{k}) \in X_{\mathcal{R},q}(\dot\Omega),\\
&\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^l F_{\mathcal{R},\lambda}\mathbf{U}(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_1}\right\}\right)\\
&\le \gamma_4(\sigma_2 + \gamma_{\sigma_2}\sigma_1 + \gamma_{\sigma_1}\gamma_{\sigma_2}\lambda_1^{-1/2}) \quad (l = 0, 1),
\end{aligned}$$

for any $\sigma_1, \sigma_2 > 0$ and for any $\lambda_1 \ge \lambda_0$. Here, and subsequently, γ_{σ_1}, γ_{σ_2} are positive constants depending on σ_1, σ_2, respectively.

Proof. We can treat $\mathcal{V}^i(\lambda)$ in the same manner as in [26, Subsection 5.4], so that we here only consider $\mathcal{P}^i(\lambda)$. Let $\mathrm{Div}(\mu\mathbf{D}(\varphi\mathbf{u})) - \varphi\,\mathrm{Div}(\mu\mathbf{D}(\mathbf{u})) =$

[4] Holomorphic property can be proved in the same manner as in [26, Proposition 5.3 (ii)].

$\mathcal{C}_1(\mu,\varphi)\nabla\mathbf{u}+\mathcal{C}_0(\mu,\varphi)\mathbf{u}$ for any scalar functions μ, φ and for any N-vector function $\mathbf{u}$, where we have set

$$\begin{aligned}&\mathcal{C}_0(\mu,\varphi)\mathbf{u} =< \nabla\mu,\mathbf{u} > \nabla\varphi + < \nabla\mu,\nabla\varphi > \mathbf{u} + \mu\left\{(\nabla^2\varphi)\mathbf{u}+(\Delta\varphi)\mathbf{u}\right\},\\ &\mathcal{C}_1(\mu,\varphi)\nabla\mathbf{u} = \mu\left\{\mathbf{D}(\mathbf{u})\nabla\varphi+(\nabla\varphi)\,\mathrm{div}\,\mathbf{u}+(\nabla\mathbf{u})\nabla\varphi\right\}.\end{aligned}$$

Analogously, let $\mathbf{D}(\varphi\mathbf{u})-\varphi\mathbf{D}(\mathbf{u})=\mathcal{D}(\nabla\varphi)\mathbf{u}$, $\mathrm{div}(\varphi\mathbf{u})-\varphi\,\mathrm{div}\,\mathbf{u}=\mathcal{E}(\nabla\varphi)\mathbf{u}$.

Each term of $\mathcal{P}^0(\lambda)(\mathbf{f},\mathbf{h},\mathbf{k})$ is rewritten as follows:

$$\begin{aligned}&(\rho^i_j)^{-1}\{\nabla K(\zeta^i_j\mathbf{u}^i_j)-\zeta^i_j\nabla K^i_j(\mathbf{u}^i_j)\}\\ &=(\rho^i_j)^{-1}\nabla\left(K(\zeta^i_j\mathbf{u}^i_j)-\zeta^i_jK^i_j(\mathbf{u}^i_j)\right)+(\rho^i_j)^{-1}(\nabla\zeta^i_j)K^i_j(\mathbf{u}^i_j).\end{aligned}$$

We start with inequalities of Poincaré type with an uniform constant, which are proved similarly to the proof of [25, Lemma 3.4, Lemma 3.5].

Lemma 5.5. *Let $1<q<\infty$. Then there exists a constant $c_1>0$, independent of $j\in\mathbf{N}$, such that*

$$\begin{aligned}&\|\varphi-c^1_j(\varphi)\|_{W^1_q(\mathcal{H}^0_j\cap B^1_j)}\le c_1\|\nabla\varphi\|_{L_{q(\mathcal{H}^0_j\cap B^1_j)}} &&\text{for any } \varphi\in\widehat{W}^1_q(\mathcal{H}^0_j),\\ &\|\psi-c^1_j(\psi)\|_{W^1_q(\Omega\cap B^1_j)}\le c_1\|\nabla\psi\|_{L_q(\Omega\cap B^1_j)} &&\text{for any } \psi\in\mathcal{W}^1_q(\Omega),\\ &\|\psi\|_{W^1_q(\Omega\cap B^2_j)}\le c_1\|\nabla\psi\|_{L_q(\Omega\cap B^2_j)} &&\text{for any } \psi\in\mathcal{W}^1_q(\Omega),\\ &\|\psi-c^i_j(\psi)\|_{W^1_q(\Omega\cap B^i_j)}\le c_1\|\nabla\psi\|_{L_q(\Omega\cap B^i_j)} &&\text{for any } \psi\in\mathcal{W}^1_q(\Omega),\ i=3,4,5.\end{aligned}$$

Here, $c^1_j(\varphi)$ and $c^i_j(\psi)$ $(i=1,3,4,5)$ are suitable constants depending on φ and ψ, respectively.

To handle $(\rho^i_j)^{-1}(\nabla\zeta^i_j)K^i_j(\mathbf{u}^i_j)$, we use the following lemma.

Lemma 5.6. *Let $1<q<\infty$. Then there exists a constant c_2, independent of $j\in\mathbf{N}$, such that*

$$\begin{aligned}\|K^1_j(\mathbf{u})\|_{L_q(\mathcal{H}^1_j\cap B^1_j)}\le c_2\Big(&\|\nabla\mathbf{u}\|_{L_q(\mathcal{H}^1_j)}+\|\nabla\mathbf{u}\|^{1-1/q}_{L_q(\mathcal{H}^1_{+j})}\|\nabla^2\mathbf{u}\|^{1/q}_{L_q(\mathcal{H}^1_{+j})}\\ &+\|\nabla\mathbf{u}\|^{1-1/q}_{L_q(\mathcal{H}^1_{-j})}\|\nabla^2\mathbf{u}\|^{1/q}_{L_q(\mathcal{H}^1_{-j})}\Big),\\ \|K^i_j(\mathbf{v})\|_{L_q(\mathcal{H}^i_j\cap B^i_j)}\le c_2\Big(&\|\nabla\mathbf{v}\|_{L_q(\mathcal{H}^i_j)}+\delta^i\|\nabla\mathbf{v}\|^{1-1/q}_{L_q(\mathcal{H}^i_j)}\|\nabla^2\mathbf{v}\|^{1/q}_{L_q(\mathcal{H}^i_j)}\Big),\end{aligned}\tag{5.19}$$

for any $\mathbf{u}\in W^2_q(\mathcal{H}^1_j)^N$ and for any $\mathbf{v}\in W^2_q(\mathcal{H}^i_j)^N$ $(i=2,\ldots,5)$, respectively, where δ^i are symbols defined by $\delta^i=1$ $(i=2,3)$ and $\delta^i=0$ $(i=4,5)$.

Remark 5.7. Applying Young's inequality to (5.19), we have

$$\|K^i_j(\mathbf{u})\|_{L_q(\mathcal{H}^i_j\cap B^i_j)} \leq \sigma_1\|\nabla^2\mathbf{u}\|_{L_q(\mathcal{H}^i_j)} + \gamma_{\sigma_1}\|\nabla\mathbf{u}\|_{L_q(\mathcal{H}^i_j)} \quad (i=1,\ldots,5), \tag{5.20}$$

for any $\sigma_1 > 0$ and for any $\mathbf{u} \in W^2_q(\mathcal{H}^i_j)^N$.

Proof of Lemma 5.6. We here show the case[5] $K^1_j(\mathbf{u})$. In the following, C stands for generic constants independent of $j \in \mathbf{N}$, and recall that $\mathcal{H}^0_j = \Phi^1_j(\mathbf{R}^N) = \mathbf{R}^N$, $\mathcal{H}^1_j = \mathcal{H}^1_{+j} \cup \mathcal{H}^1_{-j}$ ($\mathcal{H}^1_{\pm j} = \Phi^1_j(\mathbf{R}^N_\pm)$), and $\Gamma^1_j = \Phi^1_j(\mathbf{R}^N_0)$.

Let $\eta_j \in C^\infty_0(\mathcal{H}^0_j \cap B^1_j)$ in such a way that $\int_{B^1_j} \eta_j\,dx = 1$ and $\eta_j \geq 0$. Fix $\mathbf{u} \in W^2_q(\mathcal{H}^1_j)$ in what follows. Since $K^1_j(\mathbf{u}) + c$ satisfies the weak problem (5.10)-(5.11) for any constant c, we may assume that $\int_{\mathcal{H}^1_j} \eta_j K^1_j(\mathbf{u})\,dx = 0$. Given $\psi \in C^\infty_0(\mathcal{H}^0_j \cap B^1_j)$, we define a function by $\widetilde{\psi} = \psi - \eta_j \int_{B^1_j} \psi\,dx$. Then,

$$\widetilde{\psi} \in C^\infty_0(\mathcal{H}^0_j \cap B^1_j), \quad \|\widetilde{\psi}\|_{L_{q'}(\mathcal{H}^0_j)} \leq C\|\psi\|_{L_{q'}(\mathcal{H}^0_j)}, \quad \int_{\mathcal{H}^0_j\cap B^1_j} \widetilde{\psi}\,dx = 0,$$

for $q' = q/(q-1)$. These properties, combined with Lemma 5.5, yield that

$$\begin{aligned}
&|(\widetilde{\psi},\varphi)_{\mathcal{H}^0_j}| = |(\widetilde{\psi},\varphi - c^1_j(\varphi))_{\mathcal{H}^0_j\cap B^1_j}| \\
&\leq \|\widetilde{\psi}\|_{L_{q'}(\mathcal{H}^0_j\cap B^1_j)}\|\varphi - c^1_j(\varphi)\|_{L_q(\mathcal{H}^0_j\cap B^1_j)} \leq C\|\psi\|_{L_{q'}(\mathcal{H}^0_j)}\|\nabla\varphi\|_{L_q(\mathcal{H}^0_j)},
\end{aligned}$$

for any $\varphi \in \widehat{W}^1_q(\mathcal{H}^0_j)$. Thus, $\|\widetilde{\psi}\|_{\widehat{W}^{-1}_{q'}(\mathcal{H}^0_j)} \leq C\|\psi\|_{L_{q'}(\mathcal{H}^0_j)}$, where $\widehat{W}^{-1}_{q'}(\mathcal{H}^0_j)$ is the dual spaces of $\widehat{W}^1_q(\mathcal{H}^0_j)$.

Let $\widehat{W}^2_{q'}(\mathcal{H}^1_{\pm j})$ be function spaces defined as $\widehat{W}^2_{q'}(\mathcal{H}^1_{\pm j}) = \{\theta \in \widehat{W}^1_{q'}(\mathcal{H}^1_{\pm j}) : \nabla\theta \in W^1_{q'}(\mathcal{H}^1_{\pm j})^N\}$, respectively. We choose[6] a $\Psi \in \widehat{W}^2_{q'}(\mathcal{H}^1_{+j}) \cap \widehat{W}^2_{q'}(\mathcal{H}^1_{-j})$

[5] The other cases were already proved in [25, Lemma 5.6].

[6] Since $\widetilde{\psi} \in \widehat{W}^{-1}_{q'}(\mathcal{H}^0_j)$, we can construct, by the Hahn-Banach theorem, $\omega \in L_{q'}(\mathcal{H}^0_j)^N$ such that $(\widetilde{\psi},\varphi)_{\mathcal{H}^0_j} = (\omega,\nabla\varphi)_{\mathcal{H}^0_j}$ for any $\varphi \in \widehat{W}^1_q(\mathcal{H}^0_j)$ and $\|\widetilde{\psi}\|_{\widehat{W}^{-1}_{q'}(\mathcal{H}^0_j)} = \|\omega\|_{L_{q'}(\mathcal{H}^0_j)}$. Let $u \in \widehat{W}^1_{q'}(\mathcal{H}^0_j)$ be the solution of the weak elliptic transmission problem: $((\rho^1_j)^{-1}\nabla u,\nabla\varphi)_{\mathcal{H}^1_j} = (\omega,\nabla\varphi)_{\mathcal{H}^0_j}$ for any $\varphi \in \widehat{W}^1_q(\mathcal{H}^0_j)$, which possesses the estimate: $\|\nabla u\|_{L_{q'}(\mathcal{H}^0_j)} \leq C\|\omega\|_{L_{q'}(\mathcal{H}^0_j)}$. Then choosing suitable φ shows that u satisfies the following strong problem: $-(\rho^1_j)^{-1}\Delta u = \widetilde{\psi}$ in $\mathcal{H}^1_j$, $[\![(\rho^1_j)^{-1}\partial u/\partial\mathbf{n}^1_j]\!] = 0$ on Γ^1_j, $[\![u]\!] = 0$ on Γ^1_j, and also u is a unique solution to the strong problem by the unique solvability of the weak elliptic transmission problem. Thus, by the standard Fourier analysis similar to Section

satisfying the following equations:

$$-\Delta\Psi = \widetilde{\psi} \quad \text{in } \mathcal{H}^1_j, \quad [\![\frac{\partial\Psi}{\partial\mathbf{n}^1_j}]\!] = 0 \quad \text{on } \Gamma^1_j, \quad [\![\rho^1_j\Psi]\!] = 0 \quad \text{on } \Gamma^1_j, \tag{5.21}$$

and the estimate: $\|\nabla\Psi\|_{W^1_q(\mathcal{H}^1_j)} \leq C(\|\widetilde{\psi}\|_{L_{q'}(\mathcal{H}^0_j)} + \|\widetilde{\psi}\|_{\widehat{W}^{-1}_{q'}(\mathcal{H}^0_j)})$. Then, the estimate of $\widetilde{\psi}$ above furnishes that $\|\nabla\Psi\|_{W^1_{q'}(\mathcal{H}^1_j)} \leq C\|\psi\|_{L_{q'}(\mathcal{H}^0_j)}$, and furthermore, by Gauss's divergence theorem,

$$(\nabla\Psi, \nabla\theta)_{\mathcal{H}^1_j} - \Big(\frac{\partial\Psi}{\partial\mathbf{n}^1_j}, [\![\theta]\!]\Big)_{\Gamma^1_j} = (\widetilde{\psi}, \theta)_{\mathcal{H}^1_j} \quad \text{for any } \theta \in W^1_q(\mathcal{H}^1_j) + \widehat{W}^1_q(\mathcal{H}^0_j).$$

This identity allows us to see that

$$\begin{aligned}
&(K^1_j(\mathbf{u}), \psi)_{\mathcal{H}^0_j} = (K^1_j(\mathbf{u}), \widetilde{\psi})_{\mathcal{H}^0_j} = (K^1_j(\mathbf{u}), \widetilde{\psi})_{\mathcal{H}^1_j} \\
&= (\nabla K^1_j(\mathbf{u}), \nabla\Psi)_{\mathcal{H}^1_j} - \Big([\![K^1_j(\mathbf{u})]\!], \frac{\partial\Psi}{\partial\mathbf{n}^1_j}\Big)_{\Gamma^1_j} \\
&= \Big((\rho^1_j)^{-1}\nabla K^1_j(\mathbf{u}), \nabla(\rho^1_j\Psi)\Big)_{\mathcal{H}^1_j} - \Big([\![K^i_j(\mathbf{u})]\!], \frac{\partial\Psi}{\partial\mathbf{n}^1_j}\Big)_{\Gamma^1_j},
\end{aligned}$$

which, combined with $\rho^1_j\Psi \in \widehat{W}^1_{q'}(\mathcal{H}^0_j)$ as follows from $[\![\rho^1_j\Psi]\!] = 0$ on Γ^1_j, implies that, by (5.10)-(5.11),

$$\begin{aligned}
(K^1_j(\mathbf{u}), \psi)_{\mathcal{H}^0_j} &= \Big((\rho^1_j)^{-1}\,\mathrm{Div}(\nu^1_j\mathbf{D}(\mathbf{u})) - \nabla\,\mathrm{div}\,\mathbf{u}, \nabla(\rho^1_j\Psi)\Big)_{\mathcal{H}^1_j} \\
&\quad - \Big(< [\![\nu^1_j\mathbf{D}(\mathbf{u})\mathbf{n}^1_j]\!], \mathbf{n}^1_j > - [\![\mathrm{div}\,\mathbf{u}]\!], \frac{\partial\Psi}{\partial\mathbf{n}^1_j}\Big)_{\Gamma^1_j}.
\end{aligned}$$

Thus, by Gauss's divergence theorem, we have

$$\begin{aligned}
(K^1_j(\mathbf{u}), \psi)_{\mathcal{H}^0_j} &= -\Big(\nu^1_j\mathbf{D}(\mathbf{u}), \nabla^2\Psi\Big)_{\mathcal{H}^1_j} + \int_{\Gamma^1_j} [\![< \nu^1_j\mathbf{D}(\mathbf{u})\mathbf{n}^1_j, \nabla\Psi >]\!]\, d\sigma \\
&\quad + \Big(\mathrm{div}\,\mathbf{u}, \rho^1_j\Delta\Psi\Big)_{\mathcal{H}^1_j} - \int_{\Gamma^1_j} [\![< (\mathrm{div}\,\mathbf{u})\mathbf{n}^1_j, \rho^1_j\nabla\Psi >]\!]\, d\sigma \\
&\quad - \Big(< [\![\nu^1_j\mathbf{D}(\mathbf{u})\mathbf{n}^1_j]\!], \mathbf{n}^1_j > - [\![\mathrm{div}\,\mathbf{u}]\!], \frac{\partial\Psi}{\partial\mathbf{n}^1_j}\Big)_{\Gamma^1_j},
\end{aligned} \tag{5.22}$$

where $d\sigma$ denotes the surface element of Γ^1_j.

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3 and Section 4, we have $\|\nabla^2 u\|_{L_{q'}(\mathcal{H}^1_j)} \leq C\|\widetilde{\psi}\|_{L_{q'}(\mathcal{H}^0_j)}$. If we set $\Psi = (\rho^1_j)^{-1}u$, then Ψ satisfies (5.21) and the required estimate.

At this point, we introduce trace inequalities as follows: there exists a positive constant c_3, independent of $j \in \mathbf{N}$, such that

$$\|f_\pm\|_{L_q(\Gamma^1_j)} \leq c_3 \|f_\pm\|^{1-1/q}_{L_q(\mathcal{H}^1_{\pm j})} \|\nabla f_\pm\|^{1/q}_{L_q(\mathcal{H}^1_{\pm j})} \quad \text{for any } f_\pm \in W^1_q(\mathcal{H}^1_{\pm j}), \tag{5.23}$$

$$\|f\|_{L_q(\Gamma^i_j)} \leq c_3 \|f\|^{1-1/q}_{L_q(\mathcal{H}^i_j)} \|\nabla f\|^{1/q}_{L_q(\mathcal{H}^i_j)} \quad \text{for any } f \in W^1_q(\mathcal{H}^i_j),\ i = 2, 3,$$

which can be proved by Proposition 5.1 and [13, Section 4, Proposition 16.2]. These inequalities, combined with (5.22), yield that

$$\begin{aligned} |(K^1_j(\mathbf{u}^1_j), \psi)_{\mathcal{H}^0_j}| \leq C\Big(&\|\nabla \mathbf{u}\|_{L_q(\mathcal{H}^1_j)} + \|\nabla \mathbf{u}\|^{1-1/q}_{L_q(\mathcal{H}^1_{+j})} \|\nabla^2 \mathbf{u}\|^{1/q}_{L_q(\mathcal{H}^1_{+j})} \\ &+ \|\nabla \mathbf{u}\|^{1-1/q}_{L_q(\mathcal{H}^1_{-j})} \|\nabla^2 \mathbf{u}\|^{1/q}_{L_q(\mathcal{H}^1_{-j})}\Big) \|\psi\|_{L_{q'}(\mathcal{H}^0_j)}, \end{aligned}$$

which implies that the required estimate (5.19) holds. This completes the proof of the lemma. □

We consider $(\rho^i_j)^{-1}(\nabla \zeta^i_j) K^i_j(\mathbf{u}^i_j)$. By Definition 1.2, (5.20), and (5.13), together with the formulas (5.16), we have, for any $n \in \mathbf{N}$, $\{\lambda_l\}^n_{l=1} \subset \Sigma_{\varepsilon,\lambda_1}$, and $\{\mathbf{H}_l\}^n_{l=1} \subset \mathcal{X}_{\mathcal{R},q}(\dot\Omega)$,

$$\begin{aligned} &\int_0^1 \Big\| \sum_{l=1}^n r_l(u) (\nabla \zeta^i_j) K^i_j(\mathcal{S}^i_j(\lambda_l)\mathbf{H}_l) \Big\|^q_{L_q(\dot\Omega)} du \\ &\leq \{\gamma_4(\sigma_1 + \gamma_{\sigma_1}\lambda_1^{-1/2})\}^q \int_0^1 \Big\| \sum_{l=1}^n r_l(u) \mathbf{H}_l \Big\|^q_{\mathcal{X}_{\mathcal{R},q}(\dot\Omega \cap B^i_j)} du, \end{aligned} \tag{5.24}$$

for any $\sigma_1 > 0$ and for any $\lambda_1 \geq \lambda_0$. Define $\mathbf{K}^0(\lambda)\mathbf{H}$ as

$$\mathbf{K}^0(\lambda)\mathbf{H} = \sum_{i=1}^5 \sum_{j=1}^\infty (\rho^i_j)^{-1} (\nabla \zeta^i_j) K^i_j(\mathcal{S}^i_j(\lambda)\mathbf{H}) \quad \text{for } \mathbf{H} \in \mathcal{X}_{\mathcal{R},q}(\dot\Omega).$$

In the same manner as we have obtained (5.18) from (5.17), we have, by (5.24), the following properties:

$$\begin{aligned} &\mathbf{K}^0(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), L_q(\dot\Omega)^N)), \\ &\mathbf{K}^0(\lambda) F_{\mathcal{R},\lambda}(\mathbf{f}, \mathbf{h}, \mathbf{k}) = \sum_{i=1}^5 \sum_{j=1}^\infty (\rho^i_j)^{-1} (\nabla \zeta^i_j) K^i_j(\mathbf{u}^i_j), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), L_q(\dot\Omega)^N)}\Big(\Big\{\Big(\lambda \frac{d}{d\lambda}\Big)^l \mathbf{K}^0(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_1}\Big\}\Big) \leq \gamma_4(\sigma_1 + \gamma_{\sigma_1}\lambda_1^{-1/2}), \end{aligned} \tag{5.25}$$

for $l = 0, 1$ and for any $\sigma_1 > 0$, $\lambda_1 \geq \lambda_0$.

Next, we consider $(\rho_j^i)^{-1}\nabla(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K_j^i(\mathbf{u}_j^i))$. We define a function $\mathfrak{g}_j^i, \mathfrak{h}_j^i \in W_q^1(\dot\Omega)$ by

$$\begin{aligned}
(\mathfrak{g}_j^1, \mathfrak{h}_j^1) &= (< \nu_j^1(\mathcal{D}(\nabla\zeta_j^1)\mathbf{u}_j^1)\mathbf{n}_j^1, \mathbf{n}_j^1 > -\mathcal{E}(\nabla\zeta_j^1)\mathbf{u}_j^1, 0),\\
(\mathfrak{g}_j^2, \mathfrak{h}_j^2) &= (0, < \nu_j^2(\mathcal{D}(\nabla\zeta_j^2)\mathbf{u}_j^2)\mathbf{n}_j^2, \mathbf{n}_j^2 > -\mathcal{E}(\nabla\zeta_j^2)\mathbf{u}_j^2),\\
(\mathfrak{g}_j^i, \mathfrak{h}_j^i) &= (0,0) \quad (i = 3,4,5).
\end{aligned}$$

Then, $\mathfrak{g}_j^1 = K(\zeta_j^1\mathbf{u}_j^1) - \zeta_j^1 K_j^1(\mathbf{u}_j^1)$ on Γ_j^1, $\mathfrak{h}_j^2 = K(\zeta_j^2\mathbf{u}_j^2) - \zeta_j^2 K_j^2(\mathbf{u}_j^2)$ on Γ_j^2. In addition, for any $\varphi \in \mathcal{W}_{q'}^1(\Omega)$, we have

$$\begin{aligned}
&((\rho_j^i)^{-1}\nabla(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K_j^i(\mathbf{u}_j^i)), \nabla\varphi)_{\dot\Omega}\\
&= ((\rho_j^i)^{-1}\operatorname{Div}(\nu_j^i\mathbf{D}(\zeta_j^i\mathbf{u}_j^i)) - \nabla\operatorname{div}(\zeta_j^i\mathbf{u}_j^i), \nabla\varphi)_{\mathcal{H}_j^i}\\
&- ((\rho_j^i)^{-1}(\nabla\zeta_j^i)K_j^i(\mathbf{u}_j^i), \nabla\varphi)_{\mathcal{H}_j^i} + ((\rho_j^i)^{-1}\nabla K_j^i(\mathbf{u}_j^i), (\nabla\zeta_j^i)(\varphi - c_j^i(\varphi)))_{\mathcal{H}_j^i}\\
&- ((\rho_j^i)^{-1}\operatorname{Div}(\nu_j^i(\mathbf{D}(\mathbf{u}_j^i))) - \nabla\operatorname{div}\mathbf{u}_j^i, \nabla\{\zeta_j^i(\varphi - c_j^i(\varphi))\})_{\mathcal{H}_j^i},
\end{aligned}$$

where $c_j^i(\varphi)$ are constants given in Lemma 5.5 for $i = 1,3,4,5$ and $c_j^2(\varphi) = 0$. Let $\mathcal{W}_q^{-1}(\Omega)$ be the dual space of $\mathcal{W}_{q'}^1(\Omega)$ and $< \cdot,\cdot >_\Omega$ denote the duality pairing between $\mathcal{W}_q^{-1}(\Omega)$ and $\mathcal{W}_{q'}^1(\Omega)$. Thus, if we define $I_j^i \in \mathcal{W}_q^{-1}(\Omega)$ by

$$\begin{aligned}
&< I_j^i, \varphi >_\Omega = ((\rho_j^i)^{-1}\mathcal{C}_1(\nu_j^i, \zeta_j^i)\nabla\mathbf{u}_j^i + \mathcal{C}_0(\nu_j^i, \zeta_j^i)\mathbf{u}_j^i, \nabla\varphi)_{\mathcal{H}_j^i}\\
&- (\nabla\{(\nabla\zeta_j^i)\cdot\mathbf{u}_j^i\} + (\nabla\zeta_j^i)\operatorname{div}\mathbf{u}_j^i, \nabla\varphi)_{\mathcal{H}_j^i}\\
&- 2((\rho_j^i)^{-1}(\nabla\zeta_j^i)K_j^i(\mathbf{u}_j^i), \nabla\varphi)_{\mathcal{H}_j^i} - ((\rho_j^i)^{-1}(\Delta\zeta_j^i)K_j^i(\mathbf{u}_j^i), \varphi - c_j^i(\varphi))_{\mathcal{H}_j^i}\\
&+ ((\rho_j^i)^{-1}\nu_j^i\mathbf{D}(\mathbf{u}_j^i), \nabla\{(\nabla\zeta_j^i)(\varphi - c_j^i(\varphi))\})_{\mathcal{H}_j^i}\\
&- (\operatorname{div}\mathbf{u}_j^i, \operatorname{div}\{(\nabla\zeta_j^i)(\varphi - c_j^i(\varphi))\})_{\mathcal{H}_j^i} + [\mathcal{B}_j^i(\mathbf{u}_j^i), \varphi],
\end{aligned}$$

where we have set $[\mathcal{B}_j^i(\mathbf{u}_j^i), \varphi] = 0$ for $i = 2,4,5$ and

$$\begin{aligned}
[\mathcal{B}_j^1(\mathbf{u}_j^1), \varphi] &= ([\![(\rho_j^1)^{-1}K_j^1(\mathbf{u}_j^1)]\!]\mathbf{n}_j^1, (\nabla\zeta_j^1)(\varphi - c_j^1(\varphi)))_{\Gamma_j^1}\\
&\quad - ([\![(\rho_j^1)^{-1}\nu_j^1\mathbf{D}(\mathbf{u}_j^1)]\!]\mathbf{n}_j^1 - \mathbf{n}_j^1[\![\operatorname{div}\mathbf{u}_j^1]\!], (\nabla\zeta_j^1)(\varphi - c_j^1(\varphi)))_{\Gamma_j^1},\\
[\mathcal{B}_j^3(\mathbf{u}_j^3), \varphi] &= ((\rho_j^3)^{-1}K_j^3(\mathbf{u}_j^3)\mathbf{n}_j^3, (\nabla\zeta_j^3)(\varphi - c_j^3(\varphi)))_{\Gamma_j^3}\\
&\quad - ((\rho_j^3)^{-1}\nu_j^3\mathbf{D}(\mathbf{u}_j^3)\mathbf{n}_j^3 - \mathbf{n}_j^3\operatorname{div}\mathbf{u}_j^3, (\nabla\zeta_j^3)(\varphi - c_j^3(\varphi)))_{\Gamma_j^3},
\end{aligned}$$

then we have, for $i = 1,\ldots,5$,

$$((\rho_j^i)^{-1}\nabla(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K(\mathbf{u}_j^i)), \nabla\varphi)_{\dot\Omega} = < I_j^i, \varphi >_\Omega \quad \text{for any } \varphi \in \mathcal{W}_{q'}^1(\Omega).$$

Let $\mathbf{F}$ be an element of $\mathcal{L}(\mathcal{W}_q^{-1}(\Omega), L_q(\Omega)^N)$ such that, for $\theta \in \mathcal{W}_q^{-1}(\Omega)$,

$$< \theta, \varphi >_\Omega = (\mathbf{F}(\theta), \nabla\varphi)_\Omega \quad \text{for all } \varphi \in \mathcal{W}_{q'}^1(\Omega), \quad \|\mathbf{F}(\theta)\|_{L_q(\Omega)} = \|\theta\|_{\mathcal{W}_q^{-1}(\Omega)}.$$

Such a $\mathbf{F}$ can be constructed by the Hahn-Banach theorem. Since

$$((\rho_j^i)^{-1}\nabla(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K(\mathbf{u}_j^i)), \nabla\varphi)_{\dot\Omega} = (\mathbf{F}(I_j^i), \nabla\varphi)_{\dot\Omega},$$

we see that $\nabla(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K_j^i(\mathbf{u}_j^i))$ is given by

$$\nabla(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K_j^i(\mathbf{u}_j^i)) = \nabla\mathcal{K}(\mathbf{F}(I_j^i), [\![\mathfrak{g}_j^i]\!], \mathfrak{h}_j^i|_{\Gamma_+}). \tag{5.26}$$

To prove the $\mathcal{R}$-boundedness of operators associated with (5.26), we define operators $\mathcal{I}_j^i(\lambda)$ by

$$\begin{aligned}
&< \mathcal{I}_j^i(\lambda)\mathbf{H}, \varphi >_\Omega = ((\rho_j^i)^{-1}\mathcal{C}_1(\nu_j^i, \zeta_j^i)\nabla(\mathcal{S}_j^i(\lambda)\mathbf{H}) + \mathcal{C}_0(\nu_j^i, \zeta_j^i)\mathcal{S}_j^i(\lambda)\mathbf{H}, \nabla\varphi)_{\mathcal{H}_j^i} \\
&- (\nabla\{(\nabla\zeta_j^i)\cdot\mathcal{S}_j^i(\lambda)\mathbf{H}\} + (\nabla\zeta_j^i)\,\mathrm{div}(\mathcal{S}_j^i(\lambda)\mathbf{H}), \nabla\varphi)_{\mathcal{H}_j^i} \\
&- 2((\rho_j^i)^{-1}(\nabla\zeta_j^i)K_j^i(\mathcal{S}_j^i(\lambda)\mathbf{H}), \nabla\varphi)_{\mathcal{H}_j^i} \\
&- ((\rho_j^i)^{-1}(\Delta\zeta_j^i)K_j^i(\mathcal{S}_j^i(\lambda)\mathbf{H}), \varphi - c_j^i(\varphi))_{\mathcal{H}_j^i} \\
&+ ((\rho_j^i)^{-1}\nu_j^i\mathbf{D}(\mathcal{S}_j^i(\lambda)\mathbf{H}), \nabla\{(\nabla\zeta_j^i)(\varphi - c_j^i(\varphi))\})_{\mathcal{H}_j^i} \\
&- (\mathrm{div}(\mathcal{S}_j^i(\lambda)\mathbf{H}), \mathrm{div}\{(\nabla\zeta_j^i)(\varphi - c_j^i(\varphi))\})_{\mathcal{H}_j^i} + [\mathcal{B}_j^i(\mathcal{S}_j^i(\lambda)\mathbf{H}), \varphi],
\end{aligned}$$

for any $\mathbf{H} \in \mathcal{X}_{\mathcal{R},q}(\dot\Omega)$ and for any $\varphi \in \mathcal{W}_{q'}^1(\Omega)$. In addition, we define operators $\mathcal{J}_j^i(\lambda)$ by

$$\mathcal{J}_j^i(\lambda)\mathbf{H} =< \nu_j^i(\mathcal{D}(\nabla\zeta_j^i)(\mathcal{S}_j^i(\lambda)\mathbf{H}))\mathbf{n}_j^i, \mathbf{n}_j^i > - \mathcal{E}(\nabla\zeta_j^i)(\mathcal{S}_j^i(\lambda)\mathbf{H}) \quad (i = 1, 2).$$

By Lemma 5.5 and (5.23), we have

$$\begin{aligned}
|[\mathcal{B}_j^1(\mathcal{S}_j^1(\lambda)\mathbf{H}), \varphi]| \le \gamma_4\Big(&\|K_j^1(\mathcal{S}_j^1(\lambda)\mathbf{H})\|_{L_q(\mathcal{H}_{+j}^1)}^{1-1/q}\|\nabla K_j^1(\mathcal{S}_j^1(\lambda)\mathbf{H})\|_{L_q(\mathcal{H}_{+j}^1)}^{1/q} \\
&+ \|K_j^1(\mathcal{S}_j^1(\lambda)\mathbf{H})\|_{L_q(\mathcal{H}_{-j}^1)}^{1-1/q}\|\nabla K_j^1(\mathcal{S}_j^1(\lambda)\mathbf{H})\|_{L_q(\mathcal{H}_{-j}^1)}^{1/q} \\
&+ \|\nabla\mathcal{S}_j^1(\lambda)\mathbf{H}\|_{L_q(\mathcal{H}_{+j}^1)}^{1-1/q}\|\nabla^2\mathcal{S}_j^1(\lambda)\mathbf{H}\|_{L_q(\mathcal{H}_{+j}^1)}^{1/q} \\
&+ \|\nabla\mathcal{S}_j^1(\lambda)\mathbf{H}\|_{L_q(\mathcal{H}_{-j}^1)}^{1-1/q}\|\nabla^2\mathcal{S}_j^1(\lambda)\mathbf{H}\|_{L_q(\mathcal{H}_{-j}^1)}^{1/q}\Big)\|\nabla\varphi\|_{L_{q'}(\Omega\cap B_j^1)},
\end{aligned}$$

which, combined with Young's inequality and (5.20), furnishes that

$$\begin{aligned}
|[\mathcal{B}_j^1(\mathcal{S}_j^1(\lambda)\mathbf{H}), \varphi]| \le \gamma_4\Big(&(\sigma_2 + \sigma_1\gamma_{\sigma_2})\|\nabla^2\mathcal{S}_j^1(\lambda)\mathbf{H}\|_{L_q(\mathcal{H}_j^1)} \\
&+ \gamma_{\sigma_1}\gamma_{\sigma_2}\|\nabla\mathcal{S}_j^1(\lambda)\mathbf{H}\|_{L_q(\mathcal{H}_j^1)}\Big)\|\nabla\varphi\|_{L_{q'}(\Omega\cap B_j^1)},
\end{aligned}$$

for any $\sigma_1, \sigma_2 > 0$. Similarly, we can estimate $[\mathcal{B}_j^3(\mathcal{S}_j^3(\lambda)\mathbf{H}), \varphi]$. Since $[\mathcal{B}_j^i(\mathbf{u}_j^i), \varphi]$ are linear with respect to $\mathbf{u}_j^i$, the inequalities of $[\mathcal{B}_j^i(\mathcal{S}_j^i(\lambda)\mathbf{H}), \varphi]$ $(i = 1, 3)$ above yield that

$$\Big| < \sum_{l=1}^n a_l \mathcal{I}_j^i(\lambda_l)\mathbf{H}_l, \varphi >_\Omega \Big| \leq \gamma_4 \Big\{ \gamma_{\sigma_1}\gamma_{\sigma_2} \Big\| \sum_{l=1}^n a_l \mathcal{S}_j^i(\lambda_l)\mathbf{H}_l \Big\|_{W_q^1(\mathcal{H}_j^i)} \tag{5.27}$$
$$+ (\sigma_2 + \sigma_1\gamma_{\sigma_2}) \Big\| \sum_{l=1}^n a_l \nabla^2 \mathcal{S}_j^i(\lambda_l)\mathbf{H}_l \Big\|_{L_q(\mathcal{H}_j^i)} \Big\} \|\nabla\varphi\|_{L_{q'}(\Omega \cap B_j^i)},$$

with $i = 1, \ldots, 5$ for any $\varphi \in \mathcal{W}_{q'}^1(\Omega)$ and for any $n \in \mathbf{N}$, $\{a_l\}_{l=1}^n \subset \mathbf{C}$, $\{\lambda_l\}_{l=1}^n \subset \Sigma_{\varepsilon,\lambda_1}$, and $\{\mathbf{H}_l\}_{l=1}^n \subset \mathcal{X}_{\mathcal{R},q}(\dot\Omega)$. The estimate (5.27) with $n = 1$, together with (5.12) and (5.16), shows that

$$| < \mathcal{I}_j^i(\lambda)\mathbf{H}, \varphi >_\Omega | \leq M \|\mathbf{H}\|_{\mathcal{X}_{\mathcal{R},q}(\dot\Omega \cap B_j^i)} \|\nabla\varphi\|_{L_{q'}(\Omega \cap B_j^i)} \quad \text{for all } \varphi \in \mathcal{W}_{q'}^1(\Omega),$$

for any $\lambda \in \Sigma_{\varepsilon,\lambda_1}$ and $\mathbf{H} \in \mathcal{X}_{\mathcal{R},q}(\dot\Omega)$ with some positive constant M independent of $j \in \mathbf{N}$, which, combined with Lemma 5.2, furnishes that the infinite sum $\mathbf{I}^i(\lambda)\mathbf{H} = \sum_{j=1}^\infty \mathcal{I}_j^i(\lambda)\mathbf{H}$ exists in the strong topology of $\mathcal{W}_q^{-1}(\Omega)$. In addition, by (5.27) with Hölder's inequality and by Lemma 5.2 again,

$$\Big\| \sum_{l=1}^n a_l \mathbf{I}^i(\lambda_l)\mathbf{H}_l \Big\|_{\mathcal{W}_q^{-1}(\Omega)}^q \leq 2^q (\gamma_4)^q \Big\{ (\gamma_{\sigma_1}\gamma_{\sigma_2})^q \sum_{j=1}^\infty \Big\| \sum_{l=1}^n a_l \mathcal{S}_j^i(\lambda_l)\mathbf{H}_l \Big\|_{W_q^1(\mathcal{H}_j^i)}^q$$
$$+ (\sigma_2 + \sigma_1\gamma_{\sigma_2})^q \sum_{j=1}^\infty \Big\| \sum_{l=1}^n a_l \nabla^2 \mathcal{S}_j^i(\lambda_l)\mathbf{H}_l \Big\|_{L_q(\mathcal{H}_j^i)}^q \Big\}.$$

This inequality, combined with monotone convergence theorem, Proposition 2.5, and (5.13), together with the formulas (5.16), yields that, by Definition 1.2 and (5.2),

$$\int_0^1 \Big\| \sum_{l=1}^n r_l(u) \mathbf{I}^i(\lambda_l)\mathbf{H}_l \Big\|_{\mathcal{W}_q^{-1}(\Omega)}^q \, du$$
$$\leq \gamma_4 \Big((\sigma_2 + \sigma_1\gamma_{\sigma_2})^q + (\gamma_{\sigma_1}\gamma_{\sigma_2}\lambda_1^{-1/2})^q \Big) \int_0^1 \Big\| \sum_{l=1}^n r_l(u)\mathbf{H}_l \Big\|_{\mathcal{X}_{\mathcal{R},q}(\dot\Omega)}^q \, du,$$

for any $\sigma_1, \sigma_2 > 0$ and $\lambda_1 \geq \lambda_0$. Thus, setting $\mathbf{I}(\lambda)\mathbf{H} = \sum_{i=1}^5 \mathbf{I}^i(\lambda)\mathbf{H}$ and using Proposition 2.4, we have

$$\mathbf{I}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), \mathcal{W}_q^{-1}(\Omega))), \tag{5.28}$$

$$\mathbf{I}(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k}) = \sum_{i=1}^{5}\sum_{j=1}^{\infty} I_j^i,$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega),\mathcal{W}_q^{-1}(\Omega))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l \mathbf{I}(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_1}\Big\}\Big)$$
$$\leq \gamma_4(\sigma_2 + \sigma_1\gamma_{\sigma_2} + \gamma_{\sigma_1}\gamma_{\sigma_2}\lambda_1^{-1/2}) \quad (l = 0, 1).$$

Analogously, we can prove the existence of operator families $\mathbf{J}^1(\lambda), \mathbf{J}^2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega)), W_q^1(\dot\Omega))$ such that

$$\mathbf{J}^1(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k}) = \sum_{j=1}^{\infty}\mathfrak{g}_j^1, \quad \mathbf{J}^2(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k}) = \sum_{j=1}^{\infty}\mathfrak{h}_j^2, \tag{5.29}$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega),\widetilde{\mathcal{X}}_{\mathcal{R},q}(\dot\Omega))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l \big(\widetilde{F}_{\mathcal{R},\lambda}\mathbf{J}^i(\lambda)\big) : \lambda \in \Sigma_{\varepsilon,\lambda_1}\Big\}\Big)$$
$$\leq \gamma_4(\sigma_2 + \sigma_1\gamma_{\sigma_2} + \gamma_{\sigma_1}\gamma_{\sigma_2}\lambda_1^{-1/2}) \quad (i = 1, 2,\ l = 0, 1),$$

for any $\sigma_1, \sigma_2 > 0$ and for any $\lambda_1 \geq \lambda_0$, where

$$\widetilde{\mathcal{X}}_{\mathcal{R},q}(\dot\Omega) = L_q(\dot\Omega)^{N^2} \times L_q(\dot\Omega)^N \times W_q^1(\dot\Omega)^N, \quad \widetilde{F}_{\mathcal{R},\lambda}\mathbf{u} = (\nabla\mathbf{u}, \lambda^{1/2}\mathbf{u}, \mathbf{u}).$$

In view of (5.26), we define $\mathbf{L}^0(\lambda)\mathbf{H}$ as

$$\mathbf{L}^0(\lambda)\mathbf{H} = \nabla\mathcal{K}(\mathbf{F}(\mathbf{I}(\lambda)\mathbf{H}), [\![\mathbf{J}^1(\lambda)\mathbf{H}]\!], \mathbf{J}^2(\lambda)\mathbf{H}|_{\Gamma_+}) \quad \text{for } \mathbf{H} \in \mathcal{X}_{\mathcal{R},q}(\dot\Omega).$$

Then, by the continuity of $\mathcal{K}$, (5.28), (5.29), and Proposition 2.4, we see that

$$\mathbf{L}^0(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_1}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega), L_q(\dot\Omega)^N)), \tag{5.30}$$
$$\mathbf{L}^0(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k}) = \sum_{i=1}^{5}\sum_{j=1}^{\infty}\nabla\left(K(\zeta_j^i\mathbf{u}_j^i) - \zeta_j^i K_j^i(\mathbf{u}_j^i)\right),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega),L_q(\dot\Omega)^N)}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l \mathbf{L}^0(\lambda) : \lambda \in \Sigma_{\varepsilon,\lambda_1}\Big\}\Big)$$
$$\leq \gamma_4(\sigma_2 + \sigma_1\gamma_{\sigma_2} + \gamma_{\sigma_1}\gamma_{\sigma_2}\lambda_1^{-1/2}) \quad (l = 0, 1),$$

for any $\sigma_1, \sigma_2 > 0$ and $\lambda_1 \geq \lambda_0$. By (5.25), (5.29), and (5.30), we can construct the required operator $\mathbf{U}(\lambda)$ of Lemma 5.4. This completes the proof of the lemma. □

5.5. **Proof of Theorem 2.2.** In Lemma 5.4, we choose σ_1, σ_2, and λ_1 in such a way that $\gamma_4\sigma_2 < 1/8$, $\gamma_4\gamma_{\sigma_2}\sigma_1 < 1/8$, and $\gamma_4\gamma_{\sigma_1}\gamma_{\sigma_2}\lambda_1^{-1/2} < 1/4$, successively, and thus

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l F_{\mathcal{R},\lambda}\mathbf{U}(\lambda) : \lambda\in\Sigma_{\varepsilon,\lambda_1}\Big\}\Big) \le \frac{1}{2} \quad (l=0,1).$$

These inequalities imply that

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{\mathcal{R},q}(\dot\Omega))}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (I-F_{\mathcal{R},\lambda}\mathbf{U}(\lambda))^{-1} : \lambda\in\Sigma_{\varepsilon,\lambda_1}\Big\}\Big) \le 2 \quad (l=0,1).$$

Similarly to Section 4, setting $\mathbf{B}(\lambda) = \mathbf{S}(\lambda)(I - F_{\mathcal{R},\lambda}\mathbf{U}(\lambda))^{-1}$ with (5.18) yields that $\mathbf{u} = \mathbf{B}(\lambda)F_{\mathcal{R},\lambda}(\mathbf{f},\mathbf{h},\mathbf{k})$ solves the problem (2.3) and $\mathbf{B}(\lambda)$ satisfies (2.9). The uniqueness of (2.3) follows from the solvablility of the weak elliptic transmission problem on $\mathcal{W}^1_{q'}(\Omega)$ for $\rho_\pm$ and the solvability of (2.3) for q' in the same manner as in the proof of Theorem 1.6. This completes the proof of Theorem 2.2.

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