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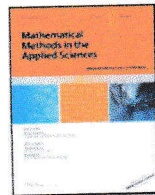
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Global well-posedness for free boundary problem of the Oldroyd-B model fluid flow

Sri Maryani^{*†}

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In this paper, we prove the global well-posedness of non-Newtonian viscous fluid flow of the Oldroyd-B model with free surface in a bounded domain of N -dimensional Euclidean space \mathbb{R}^N ($N \geq 2$). The assumption of the problem is that the initial data are small enough and orthogonal to rigid motions. Copyright © 2015 John Wiley & Sons, Ltd.

Keywords: non-Newtonian compressible viscous barotropic fluid flow; Oldroyd-B type; global well-posedness; orthogonal to rigid motion

1. Introduction

In this paper, we consider the global well-posedness of non-Newtonian compressible viscous barotropic fluid flow of Oldroyd-B type with free surface in a bounded domain, which is formulated in the following. Let Ω be a bounded domain in N -dimensional Euclidean space \mathbb{R}^N ($N \geq 2$) occupied by a compressible viscous barotropic non-Newtonian fluid of Oldroyd-B type. We assume that the boundary of Ω consists of two parts Γ and S , where $\Gamma \cap S = \emptyset$. Let Ω_t and Γ_t be time evolutions of Ω and Γ , while S be fixed. We assume that the boundary of Ω_t consists of Γ_t and S with $\Gamma_t \cap S = \emptyset$. Let $\rho : \Omega \times [0, T) \rightarrow \mathbb{R}$, $\mathbf{v} : \Omega \times [0, T) \rightarrow \mathbb{R}^N$ and $\tau : [0, \infty) \times \Omega \rightarrow \mathbb{R}^{N \times N}$ be the density field, the velocity field, and the elastic part of the stress tensor, respectively, which satisfy the system of equations:

$$\left\{ \begin{array}{ll} \partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 & \text{in } \Omega_t, \\ \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) - \operatorname{Div} \mathbf{T}(\mathbf{v}, \rho) = \beta \operatorname{Div} \tau & \text{in } \Omega_t, \\ \partial_t \tau + \mathbf{v} \cdot \nabla \tau + \gamma \tau = \delta \mathbf{D}(\mathbf{v}) + g_\alpha(\nabla \mathbf{v}, \tau) & \text{in } \Omega_t, \\ (\mathbf{T}(\mathbf{v}, \rho) + \beta \tau) \mathbf{n}_t = -P(\rho_*) \mathbf{n}_t & \text{on } \Gamma_t, \\ \mathbf{v} = 0 & \text{on } S, \\ (\rho, \mathbf{v}, \tau)|_{t=0} = (\rho_*, \mathbf{v}_0, \tau_0) & \text{in } \Omega, \\ \Omega_t|_{t=0} = \Omega_0, \quad \Gamma_t|_{t=0} = \Gamma, & \end{array} \right. \quad (1.1)$$

for $0 < t < T$. Here, ρ_* is a positive constant describing the mass density of the reference domain Ω , $\mathbf{T}(\mathbf{v}, \rho)$ is the stress tensor of the form

$$\mathbf{T}(\mathbf{v}, \rho) = \mathbf{S}(\mathbf{v}) - P(\rho) \mathbf{I} \text{ with } \mathbf{S}(\mathbf{v}) = \mu \mathbf{D}(\mathbf{v}) + (\nu - \mu) \operatorname{div} \mathbf{v} \mathbf{I}, \quad (1.2)$$

$\mathbf{D}(\mathbf{v})$, $\mathbf{v} = (v_1, \dots, v_N)$, the doubled deformation tensor whose (i, j) components are $D_{ij}(\mathbf{v}) = \partial_i v_j + \partial_j v_i$ ($\partial_i = \partial/\partial x_i$), \mathbf{I} the $N \times N$ identity matrix, μ, ν, β, γ , and δ are positive constants (μ and ν are the first and second viscosity coefficients, respectively), \mathbf{n}_t is the unit outer normal to Γ_t , $P(\rho)$ a C^∞ function defined for $\rho > 0$ that satisfies that $P'(\rho) > 0$ for $\rho > 0$. Moreover, the function $g_\alpha(\nabla \mathbf{v}, \tau)$ has a form

$$g_\alpha(\nabla \mathbf{v}, \tau) = \mathbf{W}(\mathbf{v})\tau - \tau \mathbf{W}(\mathbf{v}) + \alpha(\tau \mathbf{D}(\mathbf{v}) + \mathbf{D}(\mathbf{v})\tau), \quad (1.3)$$

where α is a constant with $-1 \leq \alpha \leq 1$ and $\mathbf{W}(\mathbf{v})$ the doubled antisymmetric part of the gradient $\nabla \mathbf{v}$ whose (i, j) components are $W_{ij}(\mathbf{v}) = \partial_i v_j - \partial_j v_i$. Finally, for any matrix field \mathbf{K} whose components are K_{ij} , the quantity $\operatorname{Div} \mathbf{K}$ is an N vector whose i th component is $\sum_{j=1}^N \partial_j K_{ij}$, and also $\operatorname{div} \mathbf{v} = \sum_{j=1}^N \partial_j v_j$, and $\mathbf{v} \cdot \nabla \mathbf{v}$ is an N vector whose i th component is $\sum_{j=1}^N v_j \partial_j v_i$.

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Aside from the dynamical system (1.1), a further kinematic condition for Γ_t is satisfied, which gives

$$\Gamma_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in \Gamma)\}, \quad (1.4)$$

where $\mathbf{x} = \mathbf{x}(\xi, t)$ is the solution to the Cauchy problem:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) \quad (t > 0), \quad \mathbf{x}|_{t=0} = \xi \in \overline{\Omega}. \quad (1.5)$$

This fact means that the free surface Γ_t consists of the same fluid particles, which do not leave it and are not incident of it from inside Ω_t for $t > 0$. It is clear that $\Omega_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in \Omega)\}$.

The set of equations in (1.1) was introduced by Oldroyd [1] in the incompressible viscous fluid case, that is, ρ is a positive constant in (1.1), to describe the flow of viscoelastic fluids, which provides a simple linear viscoelastic model for dilute polymer solutions, based on the dumbbell model. The fluid described by the set of equations in (1.1) is called the Oldroyd-B type fluid after its creator James G. Oldroyd. The mathematical study of the Oldroyd-B type fluids was started by a pioneering paper by Guillopé and Saut [2], where the local well-posedness was proved in suitable Hilbert space H^s in the case of bounded domain with non-slip boundary condition. Later, the global well-posedness for small initial data was proved by Fernández-Cara, Guillén, and Ortega [3] in the L_p space setting with small coupling constant β . The local well-posedness of the free boundary problem was proved by Nesensohn in the L_p space setting. The global well-posedness of free boundary value problem was proved by Xu, Zhang and Zhang [4] in an infinite strip and with surface tension on the upper free boundary provided that the initial data is sufficiently close to the equilibrium state in H^3 . The global well-posedness in the exterior domain with non-slip condition case was proved by Hieber, Naito, and Shibata [5] and by Fang, Hieber and Zi [6] in the H^3 space setting.

The existence of global weak solutions in the case of $\Omega = \mathbb{R}^N$ and $\alpha = 0$ in (1.3) was proved by Lions and Masmoudi in [7]. The situation of infinite Weissenburg numbers, that is, $\gamma = 0$ case, was considered for $\Omega = \mathbb{R}^3$ or for bounded domains $\Omega \subset \mathbb{R}^3$ with non-slip boundary condition that can be found in [8–10], and the result in the two-dimensional situation can be found in [11–13] in the H^s space setting. Moreover, the global well-posedness of strong L_p solutions of zero Weissenburg number problem was proved by Geissert, Nesensohn, and Shibata [14] in the bounded domain with non-slip boundary condition.

On the other hand, concerning the study for the compressible case, we know only the result about the local well-posedness of non-Newtonian compressible viscous barotropic fluid flow of Oldroyd-B type with free surface because of Maryani [15] in the maximal L_p - L_q regularity class in a bounded domain and some unbounded domains that satisfy some uniformity. This paper is the continuation of Maryani [15], and the global well-posedness of problem (1.1) is proved in the bounded domain case.

Concerning the local and global well-posedness of the free boundary problem for the Newtonian compressible viscous barotropic fluid flow, we know several results. Concerning the local well-posedness, Secchi and Valli [16] proved for Newtonian fluid flow without surface tension in the L_2 framework. While, Tani [17] investigated the same problem in Hölder spaces. Later on, similar problem with surface tension in the L_2 framework and in Hölder spaces is proved by Solonnikov and Tani [18] and Denisova and Solonnikov [19, 20], respectively.

Moreover, the global well-posedness of the free boundary problem for the Newtonian flow without surface tension was first studied by Zajackowski [21] in the L_2 framework using the energy method. One year later, the same problem with surface tension was investigated by Zajackowski [22] and Solonnikov and Tani [23] independently under the assumption that initial velocity is small enough, and initial mass density and the reference domain are close to positive constant and a ball using the energy method in the L_2 framework, respectively.

Recently, Enomoto, von Below, and Shibata [24] proved the local well-posedness in the maximal L_p - L_q regularity class in a bounded domain and some unbounded domains that satisfy some uniformity. Moreover, Shibata [25] proved the global well-posedness in a bounded domain also in the maximal L_p - L_q regularity class, assuming that the initial data are small enough and orthogonal to the rigid space. Our idea of proof follows Shibata [25].

The main aim of this study is to prove the global well-posedness for problem (1.1) in the maximal L_p - L_q regularity class in a bounded domain Ω with $2 < p < \infty$ and $N < q < \infty$, assuming that initial data are small enough and orthogonal to the rigid motion when $S = \emptyset$. To prove it, we use the Lagrange coordinate instead of the Euler coordinate and prolong the local in time solutions in the Lagrange coordinate to any time interval. To do this, the decay properties of solutions play an essential role, which is proved in the case where the velocity field is orthogonal to the rigid motion in the Euler coordinate when $S = \emptyset$. Such observation was found first by Solonnikov [26] in his energy inequality for the full nonlinear-system, but we formulate this fact in the estimates of solutions to the linearized equations.

Because Ω_t should be decided, we formulate problem (1.1) in the Lagrange coordinates. In fact, if the velocity field $\mathbf{u}(\xi, t)$ is known as a function of the Lagrange coordinates $\xi \in \Omega$, then in view of (1.5), the connection between the Euler coordinates $x \in \Omega_t$ and the Lagrange coordinates $\xi \in \Omega$ is written in the form

$$x = \xi + \int_0^t \mathbf{u}(\xi, s) ds \equiv \mathbf{X}_u(\xi, t), \quad (1.6)$$

where $\mathbf{u}(\xi, t) = (u_1(\xi, t), \dots, u_N(\xi, t)) = \mathbf{v}(\mathbf{X}_u(\xi, t), t)$. Let \mathbf{A} be the Jacobi matrix of the transformation $x = \mathbf{X}_u(\xi, t)$ whose (i, j) element is $a_{ij} = \delta_{ij} + \int_0^t \left(\frac{\partial u_i}{\partial \xi_j} \right)(\xi, s) ds$. There exists a small number σ such that \mathbf{A} is invertible, that is, $\det \mathbf{A} \neq 0$, provided that

$$\sup_{0 < t < T} \left\| \int_0^t \nabla \mathbf{u}(\cdot, s) ds \right\|_{L^\infty(\Omega)} \leq \sigma. \quad (1.7)$$

In this case, we have $\nabla_x = \mathbf{A}^{-1} \nabla_\xi = (\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla \mathbf{u}(\xi, s) ds)) \nabla_\xi$, where $\mathbf{V}(\mathbf{K})$ is an $N \times N$ matrix of C^∞ functions with respect to $\mathbf{K} = (K_{ij})$, which defined on $|\mathbf{K}| < 2\sigma$. Here, K_{ij} is the corresponding variable to $\int_0^t (\frac{\partial u_i}{\partial \xi_j})(\cdot, s) ds$. We have $\mathbf{V}_0(0) = 0$. Let \mathbf{n} be the unit outward normal to S , and then we have

$$\mathbf{n}_t = \frac{\mathbf{A}^{-1} \mathbf{n}}{|\mathbf{A}^{-1} \mathbf{n}|}. \quad (1.8)$$

Let $\rho(x, t)$, $\mathbf{v}(x, t)$, and $\tau(x, t)$ be solutions of (1.1), and let

$$\rho_* + \theta(\xi, t) = \rho(\mathbf{X}_u(\xi, t), t), \quad \mathbf{u}(\xi, t) = \mathbf{v}(\mathbf{X}_u(\xi, t), t), \quad \omega(\xi, t) = \tau(\mathbf{X}_u(\xi, t), t). \quad (1.9)$$

And then, problem (1.1) is written in the form

$$\begin{cases} \theta_t + \rho_* \operatorname{div} \mathbf{u} = f(\theta, \mathbf{u}, \omega) & \text{in } \Omega \times (0, T), \\ \rho_* \mathbf{u}_t - \operatorname{Div} \mathbf{S}(\mathbf{u}) + P'(\rho_*) \nabla \theta - \beta \operatorname{Div} \omega = \mathbf{g}(\theta, \mathbf{u}, \omega) & \text{in } \Omega \times (0, T), \\ \omega_t + \gamma \omega - \delta \mathbf{D}(\mathbf{u}) = \mathbf{L}(\theta, \mathbf{u}, \omega) & \text{in } \Omega \times (0, T), \\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I} + \beta \omega) \mathbf{n} = \mathbf{h}(\theta, \mathbf{u}, \omega) & \text{on } \Gamma \times (0, T), \\ \mathbf{u} = 0 & \text{on } S \times (0, T), \\ (\theta, \mathbf{u}, \omega)|_{t=0} = (\theta_0, \mathbf{v}_0, \tau_0) & \text{in } \Omega. \end{cases} \quad (1.10)$$

Here, f , \mathbf{g} , \mathbf{L} , and \mathbf{h} are nonlinear functions defined by

$$\begin{aligned} f(\theta, \mathbf{u}, \omega) &= -\theta \operatorname{div} \mathbf{u} - (\rho_* + \theta) \mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \\ \mathbf{g}(\theta, \mathbf{u}, \omega) &= -\theta \mathbf{u}_t + \operatorname{Div} \left(\mu \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} + (\nu - \mu) \mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \right) \\ &\quad + \mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \left(\mu \left(\mathbf{D}(\mathbf{u}) + \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \right) + (\nu - \mu) \left(\operatorname{div} \mathbf{u} + \mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \right) \mathbf{I} \right) \\ &\quad - P'(\rho_* + \theta) \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \theta + \beta \mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \omega - \nabla \left(\int_0^1 P''(\rho_* + \ell \theta) (1 - \ell) d\ell \theta^2 \right) \\ \mathbf{L}(\theta, \mathbf{u}, \omega) &= \delta \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} + g_\alpha(\nabla \mathbf{u}, \omega) + g_\alpha \left(\mathbf{V}_w \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u}, \omega \right) \\ \mathbf{h}(\theta, \mathbf{u}, \omega) &= - \left\{ \mu \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} + (\nu - \mu) \left(\mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \right) \mathbf{n} - \beta \omega \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \mathbf{n} \right. \\ &\quad - \left\{ \mu \left(\mathbf{D}(\mathbf{u}) + \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \right) + (\nu - \mu) \left(\operatorname{div} \mathbf{u} + \mathbf{V}_{\operatorname{div}} \left(\int_0^t \nabla \mathbf{u} ds \right) \nabla \mathbf{u} \right) \mathbf{I} \right\} \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \mathbf{n} \\ &\quad \left. + \left(\int_0^1 P''(\rho_* + \ell \theta) (1 - \ell) d\ell \theta^2 \right) \mathbf{n} + (P(\rho_* + \theta) - P(\rho_*)) \mathbf{V}_D \left(\int_0^t \nabla \mathbf{u} ds \right) \mathbf{n} \right. \end{aligned} \quad (1.11)$$

Here, $\mathbf{V}_D(\mathbf{K})$, $\mathbf{V}_w(\mathbf{K})$, and $\mathbf{V}_{\operatorname{div}}(\mathbf{K})$ are some matrices of C^∞ functions with respect to \mathbf{K} defined on $|\mathbf{K}| \leq \sigma$, which satisfy the null condition:

$$\mathbf{V}_D(0) = 0, \quad \mathbf{V}_w(0) = 0, \quad \mathbf{V}_{\operatorname{div}}(0) = 0.$$

To state our main results, at this stage, we introduce our notation used throughout the paper.

Notation \mathbb{N} , \mathbb{R} , and \mathbb{C} denote the sets of all natural numbers, real numbers, and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\operatorname{Sym}(\mathbb{R}^N)$ and $\operatorname{ASym}(\mathbb{R}^N)$ be the set of all $N \times N$ symmetric and anti-symmetric matrices, respectively. For $1 < q < \infty$, let $q' = q/(q-1)$, which is the dual exponent of q and satisfies $1/q + 1/q' = 1$. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \dots + \kappa_N$ and $\partial_x^\kappa = \partial_{x_1}^{\kappa_1} \dots \partial_{x_N}^{\kappa_N}$ with $x = (x_1, \dots, x_N)$. For scalar function f and N -vector of functions \mathbf{g} , we set

$$\begin{aligned} \nabla f &= (\partial_1 f, \dots, \partial_N f), \quad \nabla \mathbf{g} = (\partial_i g_j \mid i, j = 1, \dots, N), \\ \nabla^2 f &= \{\partial^\alpha f \mid |\alpha| = 2\}, \quad \nabla^2 \mathbf{g} = \{\partial^\alpha g_i \mid |\alpha| = 2, i = 1, \dots, N\}. \end{aligned}$$

For Banach spaces X and Y , $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y , and $\operatorname{Hol}(U, \mathcal{L}(X, Y))$ the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on a domain U in \mathbb{C} . For any domain D in \mathbb{R}^N and $1 \leq p, q \leq \infty$, $L_q(D)$, $W_q^m(D)$, $B_{p,q}^s(D)$, and $H_q^s(D)$ denote the usual Lebesgue space, Sobolev space, Besov space, and Bessel potential space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^m(D)}$, $\|\cdot\|_{B_{p,q}^s(D)}$, and $\|\cdot\|_{H_q^s(D)}$.

and $\|\cdot\|_{H_q^s(D)}$ denote their norms, respectively. We set $W_q^0(D) = L_q(D)$ and $W_q^s(D) = B_{q,q}^s(D)$. $C^\infty(D)$ denotes the set all C^∞ functions defined on D . $L_p((a,b),X)$, and $W_p^m((a,b),X)$ denote the usual Lebesgue space and Sobolev space of X -valued function defined on an interval (a,b) , while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W_p^m((a,b),X)}$ denote their norms, respectively. Moreover, we set

$$\|e^{\eta t}f\|_{L_p((a,b),X)} = \left(\int_a^b (e^{\eta t}\|f(t)\|_X)^p dt \right)^{1/p} \quad \text{for } 1 \leq p < \infty.$$

The d -product space of X is defined by $X^d = \{f = (f_1, \dots, f_d) \mid f_i \in X (i = 1, \dots, d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. We set

$$W_q^{m,\ell}(D) = \{(f, \mathbf{g}, \mathbf{H}) \mid f \in W_q^m(D), \mathbf{g} \in W_q^\ell(D)^N, \mathbf{H} \in W_q^m(D)^{N \times N}\},$$

$$\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{m,\ell}(\Omega)} = \|(f, \mathbf{H})\|_{W_q^m(\Omega)} + \|\mathbf{g}\|_{W_q^\ell(\Omega)}.$$

For $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, we set $\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^n a_j b_j$. For scalar functions f, g , and N -vectors of functions \mathbf{f}, \mathbf{g} , we set $(f, g)_D = \int_D fg \, dx$, $(\mathbf{f}, \mathbf{g})_D = \int_D \mathbf{f} \cdot \mathbf{g} \, dx$, $(f, g)_\Gamma = \int_\Gamma fg \, d\sigma$, $(\mathbf{f}, \mathbf{g})_\Gamma = \int_\Gamma \mathbf{f} \cdot \mathbf{g} \, d\sigma$, where σ is the surface element of Γ . For $N \times N$ matrices of functions $\mathbf{A} = (A_{ij})$ and $\mathbf{B} = (B_{ij})$, we set $(\mathbf{A}, \mathbf{B})_D = \int_D \mathbf{A} : \mathbf{B} \, dx$ and $(\mathbf{A}, \mathbf{B})_\Gamma = \int_\Gamma \mathbf{A} : \mathbf{B} \, d\sigma$, where $\mathbf{A} : \mathbf{B} \equiv \sum_{i,j=1}^N A_{ij} B_{ij}$. The letter C denotes generic constants, and the constant $C_{a,b,\dots}$ depends on a, b, \dots . The values of constants C and $C_{a,b,\dots}$ may change from line to line. We use small boldface letters, for example, \mathbf{u} to denote vector-valued functions and capital boldface letters, for example, \mathbf{H} to denote matrix-valued functions, respectively. But, we also use the Greek letters, for example, ρ, θ, τ , and ω , to denote mass densities, and elastic tensors unless the confusion may occur, although they are $N \times N$ matrices.

To state the compatibility condition for initial data θ_0, \mathbf{v}_0 , and τ_0 , we introduce the space $\mathcal{D}_{q,p}(\Omega)$ defined by

$$\mathcal{D}_{q,p}(\Omega) = \{(\theta_0, \mathbf{v}_0, \tau_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^N \times W_q^1(\Omega)^{N \times N} \mid$$

$$(\mathbf{S}(\mathbf{v}_0) - (P(\rho_*) + \theta_0) - P(\rho_*))\mathbf{I} + \beta \tau_0 \mathbf{n} = 0 \text{ on } \Gamma, \quad \mathbf{v}_0|_S = 0\}.$$
(1.13)

For the notational simplicity, we set

$$\|(\theta_0, \mathbf{v}_0, \tau_0)\|_{\mathcal{D}_{q,p}(\Omega)} = \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)}.$$

The following theorem about the local well-posedness of problem (1.10) was proved by Maryani [15].

Theorem 1.1

Let $N < q < \infty$, $2 < p < \infty$, and $R > 0$. Assume that Γ and S are $W_q^{2-1/q}$ compact hyper-surfaces. Then, there exists a time $T = T(R) > 0$ such that for any initial data $(\theta_0, \mathbf{v}_0, \tau_0) \in \mathcal{D}_{q,p}(\Omega)$ satisfying the conditions

$$\frac{2}{3}\rho_* < \rho_* + \theta_0(x) < \frac{4}{3}\rho_* \quad (x \in \Omega)$$
(1.14)

and

$$\|(\theta_0, \mathbf{v}_0, \tau_0)\|_{\mathcal{D}_{q,p}(\Omega)} \leq R$$
(1.15)

problem of (1.10) admits a unique solution $(\theta, \mathbf{u}, \omega)$ with

$$\theta \in W_p^1((0, T), W_q^1(\Omega)), \quad \mathbf{u} \in W_p^1((0, T), L_q(\Omega)) \cap L_p((0, T), W_q^2(\Omega)), \quad \omega \in W_p^1((0, T), W_q^1(\Omega))$$

satisfying (1.7), the range condition: $\frac{1}{3}\rho_* < \rho_* + \theta(x, t) < \frac{5}{3}\rho_*$ for any $(x, t) \in \Omega \times (0, T)$ and possessing the estimate:

$$\|\theta\|_{W_p^1((0,t),W_q^1(\Omega))} + \|\mathbf{u}\|_{W_p^1((0,t),L_q(\Omega))} + \|\mathbf{u}\|_{L_p((0,t),W_q^2(\Omega))} + \|\omega\|_{W_p^1((0,t),W_q^1(\Omega))} \leq C(R)$$

Remark 1.1

1. The range condition (1.14) follows from $\|\theta_0\|_{L_\infty(\Omega)} \leq \frac{\rho_*}{3}$.
2. The local well-posedness was proved under the assumption that Ω is a uniform $W_q^{2-1/q}$ domain in [15]. And, if Γ and S are compact $W_q^{2-1/q}$ hyper-surfaces, then Ω is a uniform $W_q^{2-1/q}$ domain.
3. By using the uniqueness of solutions, we see that if $\tau_0(x) \in \text{Sym}(\mathbb{R}^N)$ for almost all $x \in \Omega$, then $\omega(x, t) \in \text{Sym}(\mathbb{R}^N)$ for almost all $(x, t) \in \Omega \times (0, \infty)$, too.

In order to state the global well-posedness of problem (1.10), we introduce the rigid space \mathcal{R}_d , which is defined by

$$\mathcal{R}_d = \{\mathbf{A}\mathbf{x} + \mathbf{b} \mid \mathbf{A} \in \text{ASym}(\mathbb{R}^N) \text{ and } \mathbf{b} \in \mathbb{R}^N\}. \quad (1.16)$$

Let $\{\mathbf{p}_\ell\}_{\ell=1}^M$ be the system of orthonormal basis of \mathcal{R}_d .

The following theorem is our main result concerning the global well-posedness of problem (1.10).

Theorem 1.3

Let $N < q < \infty$ and $2 < p < \infty$. Let ℓ_b be a number such that $\ell_b = 3$ when $S \neq \emptyset$ and $\ell_b = 2$ when $S = \emptyset$. Assume that S and Γ are $W_q^{\ell_b-1/q}$ compact hyper-surfaces and that $\Gamma \neq \emptyset$. Assume that the viscosity coefficients μ and ν satisfy the stability condition:

$$\mu > 0, \quad \nu > \frac{N-2}{N}\mu \quad (1.17)$$

Then, there exist positive numbers ϵ and η such that for any initial data $(\theta_0, \mathbf{v}_0, \tau_0) \in \mathcal{D}_{q,p}(\Omega)$ satisfying the condition that $\tau_0(x) \in \text{Sym}(\mathbb{R}^N)$ for any $x \in \Omega$, the smallness condition: $\|(\theta_0, \mathbf{v}_0, \tau_0)\|_{\mathcal{D}_{q,p}(\Omega)} \leq \epsilon$ and the orthogonal condition:

$$((\rho_* + \theta_0)\mathbf{v}_0, \mathbf{p}_\ell)_\Omega = 0 \text{ for } \ell = 1, \dots, M \text{ when } S = \emptyset, \quad (1.18)$$

problem (1.10) with $T = \infty$ admits unique solutions θ, \mathbf{u} , and ω with

$$\theta \in W_p^1((0, \infty), W_q^1(\Omega)), \quad \mathbf{u} \in L_p((0, \infty), W_q^2(\Omega)^N) \cap W_p((0, \infty), L_q(\Omega)^N), \quad \omega \in W_p^1((0, \infty), W_q^1(\Omega)).$$

Moreover, there exists a positive constant γ_0 such that $(\theta, \mathbf{u}, \omega)$ satisfies the estimate:

$$\begin{aligned} & \|e^{\gamma s}(\partial_s \theta, \theta)\|_{L_p((0,t), W_q^1(\Omega))} + \|e^{\gamma s} \partial_s \mathbf{u}\|_{L_p((0,t), L_q(\Omega))} + \|e^{\gamma s} \mathbf{u}\|_{L_p((0,t), W_q^2(\Omega))} \\ & + \|e^{\gamma s}(\partial_s \omega, \omega)\|_{L_p((0,t), W_q^1(\Omega))} \leq C_\gamma \epsilon \end{aligned}$$

for any $t > 0$ and $\gamma \in (0, \gamma_0)$ with some positive number C_γ independent of ϵ and t .

Remark 1.4

Using the argumentation due to Ströhmer [27], we see that the map $x = \mathbf{X}_u(\xi, t)$ is bijective from Ω onto $\Omega_t = \{x = \mathbf{X}_u(\xi, t) \mid \xi \in \Omega\}$ with suitable regularity. Therefore, from Theorem 1.3, we have the global well-posedness for problem (1.1).

2. Some decay properties of solutions to the linearized problem

Let Ω be a bounded domain and let both of its boundaries S and Γ be $W_r^{2-1/r}$ hyper-surfaces with $N < r < \infty$, and let q be an exponent such that $1 < q < \infty$ and $\max(q, q') \leq r$. In this section, we show some exponential stability of solutions to the following problem:

$$\begin{cases} \partial_t \theta + \rho_* \text{div } \mathbf{u} = f & \text{in } \Omega \times (0, T), \\ \rho_* \partial_t \mathbf{u} - \text{Div } \mathbf{S}(\mathbf{u}) + P'(\rho_*) \nabla \theta - \beta \text{Div } \tau = \mathbf{g} & \text{in } \Omega \times (0, T), \\ \partial_t \tau + \gamma \tau - \delta \mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega \times (0, T), \\ \mathbf{u} = 0 & \text{on } S \times (0, T), \\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I} + \beta \tau) \mathbf{n} = \mathbf{k} & \text{on } \Gamma \times (0, T), \\ (\theta, \mathbf{u}, \tau)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega. \end{cases} \quad (2.1)$$

For this purpose, first, we analyze the corresponding generalized resolvent problem:

$$\begin{cases} \lambda \theta + \rho_* \text{div } \mathbf{u} = f & \text{in } \Omega, \\ \rho_* \lambda \mathbf{u} - \text{Div } \mathbf{S}(\mathbf{u}) + P'(\rho_*) \nabla \theta - \beta \text{Div } \tau = \mathbf{g} & \text{in } \Omega, \\ \lambda \tau + \gamma \tau - \delta \mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } S, \\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I} + \beta \tau) \mathbf{n} = \mathbf{k} & \text{on } \Gamma. \end{cases} \quad (2.2)$$

To quote some results due to Maryani [15], we introduce the \mathcal{R} -boundedness of operator families and the Weis operator valued Fourier multiplier theorem.

Definition 2.1

A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called \mathcal{R} -bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$ -valued random variables on $[0, 1]$, we have the inequality:

$$\left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) T_j x_j \right\|_Y^p du \right\}^{1/p} \leq C \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(u) x_j \right\|_X^p du \right\}^{1/p}.$$

The smallest such C is called \mathcal{R} -bounded of \mathcal{T} , which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X -valued C^∞ functions having compact support and the Schwartz space of rapidly decreasing X -valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,loc}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \rightarrow \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M \phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)). \quad (2.3)$$

The following theorem is obtained by Weis [28].

Theorem 2.2

Let X and Y be two UMD Banach spaces[‡] and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that

$$\mathcal{R}_{\mathcal{L}(X,Y)} \left(\left\{ \left(\tau \frac{d}{d\tau} \right)^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\} \right\} \right) \leq \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ . Then, the operator T_M defined in (2.3) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M , we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R}, X), L_p(\mathbb{R}, Y))} \leq C\kappa$$

for some positive constant C depending on p, X , and Y .

Remark 2.3

For the definition of UMD space, we refer to a book because of Amann [29]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

The resolvent parameter λ in problem (2.2) varies in $\Sigma_{\epsilon, \lambda_0}$ with $\Sigma_{\epsilon, \lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \pi - \epsilon, |\lambda| \geq \lambda_0\}$ ($\epsilon \in (0, \pi/2), \lambda_0 > 0$). To quote some unique existence theorem for problem (2.1), we introduce the space $\mathbf{W}_q^{-1}(\Omega)$. Let ι be the extension map $\iota : L_{1,loc}(\Omega) \rightarrow L_{1,loc}(\mathbb{R}^N)$ having the following properties :

1. For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\iota f \in W_q^1(\mathbb{R}^N)$, $\iota f = f$ in Ω and $\|\iota f\|_{W_q^1(\mathbb{R}^N)} \leq C\|f\|_{W_q^1(\Omega)}$ for $i = 0, 1$ with some constant C depending on q, r , and Ω .
2. For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\|\iota(\nabla f)\|_{H_q^{-1}(\mathbb{R}^N)} \leq C\|f\|_{L_q(\Omega)}$ with some constant C depending on q, r , and Ω .

Then, $\mathbf{W}_q^{-1}(\Omega)$ is defined by

$$\mathbf{W}_q^{-1}(\Omega) = \{f \in L_{1,loc}(\Omega) \mid \|f\|_{\mathbf{W}_q^{-1}(\Omega)} = \|\iota f\|_{H_q^{-1}(\mathbb{R}^N)} < \infty\}.$$

According to Maryani [15], we have

Theorem 2.4

Let $1 < q < \infty, 0 < \epsilon < \pi/2$, and $N < r < \infty$. Assume that $r \geq \max(q, q')$. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{2-1/r}$ compact hyper-surfaces. Let

$$\Sigma_{\epsilon, \lambda_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon, |\lambda| \geq \lambda_0\}.$$

Let

$$\begin{aligned} X_q(\Omega) &= \{(f, \mathbf{g}, \mathbf{H}, \mathbf{k}) \mid (f, \mathbf{g}, \mathbf{H}) \in W_q^{1,0}(\Omega), \mathbf{k} \in W_q^1(\Omega)^N\}, \\ \mathcal{X}_q(\Omega) &= \{(F_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5) \mid \\ &\quad F_1 \in W_q^1(\Omega), \mathbf{F}_2 \in L_q(\Omega)^N, \mathbf{F}_3 \in L_q(\Omega)^N, \mathbf{F}_4 \in L_q(\Omega)^{N^2}, \mathbf{F}_5 \in W_q^1(\Omega)^{N^2}\}. \end{aligned}$$

Then, there exists a $\lambda_0 \geq 1$ and an operator family $R(\lambda) \in \text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,2}(\Omega)))$ such that for any $(f, \mathbf{g}, \mathbf{H}, \mathbf{k}) \in X_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon, \lambda_0}$, $(\rho, \mathbf{u}, \tau) = R(\lambda)(f, \mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla \mathbf{H})$ is a unique solution to problem (2.2).

Moreover, there exists a constant C such that

$$\begin{aligned} \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau \partial \tau)^\ell (\lambda R(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq C \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau \partial \tau)^\ell (\gamma R(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq C \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^2})}(\{(\tau \partial \tau)^\ell (\lambda^{1/2} \nabla P_\nu R(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq C \quad (\ell = 0, 1), \\ \mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^3})}(\{(\tau \partial \tau)^\ell (\nabla^2 P_\nu R(\lambda)) \mid \lambda \in \Sigma_{\epsilon, \lambda_0}\}) &\leq C \quad (\ell = 0, 1), \end{aligned} \quad (2.4)$$

with $\lambda = \gamma + i\tau$. Here, P_ν is the projection operator defined by $P_\nu(\rho, \mathbf{u}, \tau) = \mathbf{u}$.

[‡]A Banach space X is called a UMD space if the Hilbert transform of X is bounded on $L_p(B\mathbb{R}, X)$ for some $p \in (1, \infty)$.

Remark 2.5

1. The F_1, F_2, F_3, F_4 , and F_5 are variables corresponding to $f, \mathbf{g}, \lambda^{1/2} \mathbf{k}, \nabla \mathbf{k}$, and \mathbf{H} , respectively.
2. Theorem 2.4 was proved in [15], where the same problem was treated even in the unbounded domain case.

As was shown in [15], applying Theorem 2.4 with the help of Theorem 2.2, we have

Theorem 2.6

Let $1 < p, q < \infty, N < r < \infty$, and $T > 0$. Assume that $\max(q, q') \leq r$. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{2-1/r}$ compact hyper-surfaces. Then, there exists a positive number η_0 such that for any initial data $(\theta_0, \mathbf{u}_0, \tau_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^N \times W_q^1(\Omega)^{N \times N}$ and right-hand sides $f, \mathbf{g}, \mathbf{H}$, and \mathbf{k} with

$$(f, \mathbf{g}, \mathbf{H}) \in L_p((0, T), W_q^{1,0}(\Omega)), \mathbf{k} \in L_p((0, T), W_q^1(\Omega)^N) \cap W_p^1((0, T), W_q^{-1}(\Omega)^N) \quad (2.5)$$

satisfying the compatibility condition:

$$(\mathbf{S}(\mathbf{u}_0) - P'(\rho_*)\theta_0 \mathbf{I} + \beta \tau_0) \mathbf{n} = \mathbf{k}|_{t=0} \text{ on } \Gamma, \quad \mathbf{u}_0 = 0 \text{ on } S. \quad (2.6)$$

Problem (2.1) admits unique solutions θ, \mathbf{u} , and τ with

$$\theta \in W_p^1((0, T), W_q^1(\Omega)), \mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, T), L_q(\Omega)^N), \tau \in W_p^1((0, T), W_q^1(\Omega)^{N \times N})$$

possessing the estimate

$$\begin{aligned} & \|\theta\|_{W_p^1((0,t), W_q^1(\Omega))} + \|\partial_s \mathbf{u}\|_{L_p((0,t), L_q(\Omega))} + \|\mathbf{u}\|_{L_p((0,t), W_q^2(\Omega))} + \|\tau\|_{W_p^1((0,t), W_q^1(\Omega))} \\ & \leq C_\gamma e^{\eta_0 t} \left\{ \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \right. \\ & \quad \left. + \|(f, \mathbf{H}, \mathbf{k})\|_{L_p((0,t), W_q^1(\Omega))} + \|\mathbf{g}\|_{L_p((0,t), L_q(\Omega))} + \|\partial_s \mathbf{k}\|_{L_p((0,t), W_q^{-1}(\Omega))} \right\} \end{aligned}$$

for any $t \in (0, T)$ with some constant C independent of t .

To prove the global well-posedness of problem (1.10), we need some decay properties of solutions to (2.1), which is stated as follows:

Theorem 2.7

Let $1 < p, q < \infty, N < r < \infty$ and $T > 0$. Assume that $\max(q, q') \leq r$. Let ℓ_b be the number defined in Theorem 1.1. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{\ell_b-1/r}$ compact hyper-surfaces. Then, for any initial data $(\theta_0, \mathbf{u}_0, \tau_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^N \times W_q^1(\Omega)^{N \times N}$ and right-hand sides $f, \mathbf{g}, \mathbf{H}$, and \mathbf{k} satisfying (2.5), the compatibility condition (2.6), and the symmetric condition: $\tau_0(x) \in \text{Sym}(\mathbb{R}^N)$ for almost all $x \in \Omega$, problem (2.1) admits unique solutions θ, \mathbf{u} , and τ with

$$\theta \in W_p^1((0, T), W_q^1(\Omega)), \mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, T), L_q(\Omega)^N), \tau \in W_p^1((0, T), W_q^1(\Omega)^{N \times N})$$

possessing the estimate:

$$\begin{aligned} & \|e^{\eta_1 s} \theta\|_{W_p^1((0,t), W_q^1(\Omega))} + \|e^{\eta_1 s} \partial_s \mathbf{u}\|_{L_p((0,t), L_q(\Omega))} + \|e^{\eta_1 s} \mathbf{u}\|_{L_p((0,t), W_q^2(\Omega))} + \|e^{\eta_1 s} \tau\|_{W_p^1((0,t), W_q^1(\Omega))} \\ & \leq C \left\{ \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} + \|e^{\eta_1 s} (f, \mathbf{g}, \mathbf{H})\|_{L_p((0,t), W_q^{1,0}(\Omega))} \right. \\ & \quad \left. + \|e^{\eta_1 s} \mathbf{k}\|_{L_p((0,t), W_q^1(\Omega))} + \|e^{\eta_1 s} \partial_s \mathbf{k}\|_{L_p((0,t), W_q^{-1}(\Omega))} + d(S) \sum_{\ell=1}^M \left(\int_0^t (e^{\eta_1 s} |\mathbf{u}(\cdot, s), \mathbf{p}_\ell|_{\Omega})^p ds \right)^{\frac{1}{p}} \right\} \end{aligned} \quad (2.7)$$

for any $t \in (0, T)$ with some positive constants C and η_1 . Here, $d(S)$ is the number such that $d(S) = 1$ when $S = \emptyset$ and $d(S) = 0$ when $S \neq \emptyset$.

Remark 2.8

The symmetric condition: $\tau_0(x) \in \text{Sym}(\mathbb{R}^N)$ for almost all $x \in \Omega$ implies that $\tau(x, t) \in \text{Sym}(\mathbb{R}^N)$ for almost all $(x, t) \in \Omega \times (0, T)$.

To prove Theorem 2.7, first, we consider problem (2.1) with $f = 0, \mathbf{g} = 0, \mathbf{H} = 0$, and $\mathbf{k} = 0$. And then, the corresponding resolvent equation is as follows:

$$\begin{cases} \lambda \theta + \rho_* \text{div } \mathbf{u} = f & \text{in } \Omega, \\ \rho_* \lambda \mathbf{u} - \text{Div } \mathbf{S}(\mathbf{u}) + P'(\rho_*) \nabla \theta - \beta \text{Div } \tau = \mathbf{g} & \text{in } \Omega, \\ \lambda \tau + \gamma \tau - \delta \mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } S, \\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I} + \beta \tau) \mathbf{n} = 0 & \text{on } \Gamma, \end{cases} \quad (2.8)$$

where θ_0, \mathbf{u}_0 , and τ_0 have been renamed f, \mathbf{g} , and \mathbf{H} , respectively. We consider problem (2.8) on the underlying space $\mathcal{H}_q(\Omega)$, which is the set of all $(f, \mathbf{g}, \mathbf{H}) \in W_q^{1,0}(\Omega)$ such that \mathbf{g} satisfies the orthogonal condition:

$$(\mathbf{g}, \mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1, \dots, M) \quad (2.9)$$

when $S = \emptyset$. Note that any solution $(\theta, \mathbf{u}, \tau)$ of problem (2.8) satisfies the orthogonal condition:

$$(\mathbf{u}, \mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1, \dots, M) \quad (2.10)$$

when $S = \emptyset$. In fact, by the divergence theorem of Gauß, we have

$$\begin{aligned} \rho_* \lambda (\mathbf{u}, \mathbf{p}_\ell)_\Omega &= (\mathbf{g}, \mathbf{p}_\ell)_\Omega + (\mathbf{k}, \mathbf{p}_\ell)_\Gamma - \frac{\mu}{2} (\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{p}_\ell))_\Omega \\ &\quad - ((v - \mu) \operatorname{div} \mathbf{u} - P'(\rho_*) \theta, \operatorname{div} \mathbf{p}_\ell)_\Omega - \frac{\beta}{2} (\tau, \mathbf{D}(\mathbf{p}_\ell))_\Omega. \end{aligned}$$

Because it holds that

$$\mathbf{D}(\mathbf{p}_\ell) = 0, \quad \operatorname{div} \mathbf{p}_\ell = 0 \quad (\ell = 1, \dots, M), \quad (2.11)$$

(2.9) implies (2.10).

Let $W_q^2(\Omega)^N$ be the set of all $\mathbf{u} \in W_q^2(\Omega)^N$, which satisfies (2.10). And also, we introduce an operator \mathcal{A} and a space $\mathcal{D}_q(\mathcal{A})$ by

$$\begin{aligned} \mathcal{A}(\theta, \mathbf{u}, \tau) &= (-\rho_* \operatorname{div} \mathbf{u}, \rho_*^{-1} (\operatorname{Div} \mathbf{S}(\mathbf{u}) - P'(\rho_*) \nabla \theta + \beta \operatorname{Div} \tau), -\gamma \tau + \delta \mathbf{D}(\mathbf{u})) \text{ for } (\theta, \mathbf{u}, \tau) \in \mathcal{D}_q(\mathcal{A}), \\ \mathcal{D}_q(\mathcal{A}) &= \{(\theta, \mathbf{u}, \tau) \in \mathcal{H}_q(\Omega) \mid \mathbf{u} \in W_q^2(\Omega)^N, \mathbf{u}|_S = 0, (\mathbf{S}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I} + \beta \tau) \mathbf{n}|_\Gamma = 0\}. \end{aligned}$$

By using \mathcal{A} , problem (2.1) with $f = 0, \mathbf{g} = 0, \mathbf{H} = 0$, and $\mathbf{k} = 0$ is written in the form:

$$\partial_t(\theta, \mathbf{u}, \tau) - \mathcal{A}(\theta, \mathbf{u}, \tau) = (0, 0, 0) \text{ for } t > 0, \quad (\theta, \mathbf{u}, \tau)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0). \quad (2.12)$$

Because \mathcal{R} boundedness implies the usual boundedness by choosing $n = 1$ in Definition 2.1, for any $\epsilon \in (0, \pi/2)$, there exists a constant $\lambda_1 > 0$ such that for any $\lambda \in \Sigma_{\epsilon, \lambda_1}$ and $(f, \mathbf{g}, \mathbf{H}) \in \mathcal{H}_q(\Omega)$, problem (2.8) admits a unique solution $(\theta, \mathbf{u}, \tau) \in \mathcal{D}_q(\Omega)$ possessing the estimate:

$$|\lambda| \|(\theta, \mathbf{u}, \tau)\|_{W_q^{1,0}(\Omega)} + \|\mathbf{u}\|_{W_q^2(\Omega)} \leq C \|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} \quad (2.13)$$

for any $\lambda \in \Sigma_{\epsilon, \lambda_1}$. Here, we used the fact that (2.9) implies (2.10).

By (2.13), we know that there exists a continuous semigroup $\{T(t)\}_{t \geq 0}$ on $\mathcal{H}_q(\Omega)$ associated with problem (2.12) that is analytic. To prove the exponential stability of $\{T(t)\}_{t \geq 0}$, it is sufficient to prove

Theorem 2.9

Let $1 < q < \infty, N < r < \infty$, and $\lambda_1 > 0$. Assume that $\max(q, q') \leq r$. Let ℓ_b be the number given in Theorem 1.1 and let λ_1 be the number given above. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{\ell_b-1/r}$ compact hyper-surfaces. Assume that

$$\mu > 0, \quad v > \frac{N-2}{N} \mu. \quad (2.14)$$

Then, for any $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq 0$ and $|\lambda| \leq \lambda_1$ and $(f, \mathbf{g}, \mathbf{H}) \in \mathcal{H}_q(\Omega)$, problem (2.2) with $\mathbf{k} = 0$ admits a unique solution $(\theta, \mathbf{u}, \tau) \in \mathcal{D}_q(\mathcal{A})$ possessing the estimate:

$$\|(\theta, \mathbf{u}, \tau)\|_{W_q^{1,2}(\Omega)} \leq C \|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)}. \quad (2.15)$$

We postpone the proof of Theorem 2.9 to Section 3. By Theorem 2.9, we have

Corollary 2.10

Let $1 < q < \infty, N < r < \infty$ and $\lambda_1 > 0$. Assume that $\max(q, q') \leq r$. Let ℓ_b be the number given in Theorem 1.1. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{\ell_b-1/r}$ compact hyper-surfaces. Assume the condition (2.14) holds. Then, the semigroup $\{T(t)\}_{t \geq 0}$ is exponentially stable on $\mathcal{H}_q(\Omega)$, that is,

$$\|T(t)(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} \leq C e^{-\eta_1 t} \|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} \quad (2.16)$$

for any $(f, \mathbf{g}, \mathbf{H}) \in \mathcal{H}_q(\Omega)$ and $t > 0$ with some positive constants C and η_1 .

Now, we are in position to prove Theorem 2.7. To reduce the problem to the semigroup setting, first, we consider the time shifted equations:

$$\left\{ \begin{array}{ll} \partial_t \theta + \lambda_0 \theta + \rho_* \operatorname{div} \mathbf{u} = f & \text{in } \Omega \times (0, T), \\ \rho_* (\partial_t \mathbf{u} + \lambda_0 \mathbf{u}) - \operatorname{Div} \mathbf{S}(\mathbf{u}) + P'(\rho_*) \nabla \theta - \beta \operatorname{Div} \tau = \mathbf{g} & \text{in } \Omega \times (0, T), \\ \partial_t \tau + \gamma \tau + \lambda_0 \tau - \delta \mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega \times (0, T), \\ \mathbf{u} = 0 & \text{on } S \times (0, T), \\ (\mathbf{S}(\mathbf{u} - P'(\rho_*) \theta \mathbf{I} + \beta \tau) \cdot \mathbf{n} = \mathbf{k} & \text{on } \Gamma_1 \times (0, T), \\ (\theta, \mathbf{u}, \tau)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega, \end{array} \right. \quad (2.17)$$

with large $\lambda_0 > 0$. For example, in the case $(\theta_0, \mathbf{u}_0, \tau_0) = (0, 0, 0)$, by using the \mathcal{R} -bounded solution operators $R(\lambda)$ given in Theorem 2.4, the solutions of (2.17) is written by the Laplace inverse transform of $R(\lambda + \lambda_0)(\hat{f}(\lambda), \hat{\mathbf{g}}(\lambda), \hat{\mathbf{H}}(\lambda))$, where $\hat{f}(\lambda), \hat{\mathbf{g}}(\lambda)$, and $\hat{\mathbf{H}}(\lambda)$ denote the Laplace transform of f, \mathbf{g} , and \mathbf{H} with respect to time variable t . Thus, using Theorem 2.4 with the help of Theorem 2.2 and employing the same argumentation as in Section 4 of Shibata [30], we have

Theorem 2.11

Let $1 < p, q < \infty, N < r < \infty, \max(q, q') \leq r$, and $T > 0$. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{2-1/r}$ compact hyper-surfaces. Then, for any initial data $(\theta_0, \mathbf{u}_0, \tau_0) \in W_q^1(\Omega) \times B_{q,p}^{2(1-1/p)}(\Omega)^N \times W_q^1(\Omega)^{N \times N}$ and right-hand sides $f, \mathbf{g}, \mathbf{H}$, and \mathbf{k} satisfying (2.5) and (2.6), problem (2.17) admits a unique solution $(\theta, \mathbf{u}, \tau)$ with

$$\theta \in W_p^1((0, T), W_q^1(\Omega)), \mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, T), L_q(\Omega)^N), \tau \in W_p^1((0, T), W_q^1(\Omega)^{N \times N}).$$

Moreover, there exists a positive constant η_2 such that θ, \mathbf{u} , and τ possess the estimate:

$$\begin{aligned} & \|e^{\eta s} \theta\|_{W_p^1((0, T), W_q^1(\Omega))} + \|e^{\eta s} \partial_s \mathbf{u}\|_{L_p((0, T), L_q(\Omega)^N)} + \|e^{\eta s} \mathbf{u}\|_{L_p((0, T), W_q^2(\Omega)^N)} + \|e^{\eta s} \tau\|_{W_p^1((0, T), W_q^1(\Omega)^{N \times N})} \\ & \leq C \left\{ \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \right. \\ & \quad \left. + \|e^{\eta s} (f, \mathbf{g}, \mathbf{H})\|_{L_p((0, T), W_q^{1,0}(\Omega))} + \|e^{\eta s} \mathbf{k}\|_{L_p((0, T), W_q^1(\Omega))} + \|e^{\eta s} \partial_s \mathbf{k}\|_{L_p((0, T), W_q^{-1}(\Omega))} \right\} \end{aligned} \quad (2.18)$$

for any $\eta \in (0, \eta_2]$ with some positive constant C depending on η_2 but independent of T .

Under the aforementioned preparations, we finish proving Theorem 2.7. We look for a solution $(\theta, \mathbf{u}, \tau)$ of the form $\theta = \kappa + \omega, \mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\tau = \psi + \varphi$ where $(\kappa, \mathbf{v}, \psi)$ and $(\omega, \mathbf{w}, \varphi)$ are solutions to the following problems:

$$\begin{cases} \partial_t \kappa + \lambda_0 \kappa + \rho_* \operatorname{div} \mathbf{v} = f & \text{in } \Omega \times (0, T), \\ \rho_* (\partial_t \mathbf{v} + \lambda_0 \mathbf{v}) - \operatorname{Div} S(\mathbf{v}) + \nabla (P'(\rho_*) \kappa) - \beta \operatorname{Div} \psi = \mathbf{g} & \text{in } \Omega \times (0, T), \\ \partial_t \psi + \lambda_0 \psi + \gamma \psi - \delta D(\mathbf{v}) = \mathbf{H} & \text{in } \Omega \times (0, T), \\ \mathbf{v} = 0 & \text{in } S \times (0, T), \\ (S(\mathbf{v}) - P'(\rho_*) \kappa \mathbf{I} + \beta \psi) \cdot \mathbf{n} = \mathbf{k} & \text{in } \Gamma \times (0, T), \\ (\kappa, \mathbf{v}, \psi)|_{t=0} = (\theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega, \end{cases} \quad (2.19)$$

$$\begin{cases} \partial_t \omega + \rho_* \operatorname{div} \mathbf{w} = \lambda_0 \kappa & \text{in } \Omega \times (0, T), \\ \rho_* \partial_t \mathbf{w} - \operatorname{Div} S(\mathbf{w}) + \nabla (P'(\rho_*) \omega) - \beta \operatorname{Div} \varphi = \rho_* \lambda_0 \mathbf{v} & \text{in } \Omega \times (0, T), \\ \partial_t \varphi + \gamma \varphi - \delta D(\mathbf{w}) = \lambda_0 \psi & \text{in } \Omega \times (0, T), \\ \mathbf{w} = 0 & \text{in } S \times (0, T), \\ (S(\mathbf{w}) - P'(\rho_*) \omega \mathbf{I} + \beta \varphi) \cdot \mathbf{n} = 0 & \text{in } \Gamma \times (0, T), \\ (\omega, \mathbf{w}, \varphi)|_{t=0} = (0, 0, 0) & \text{in } \Omega, \end{cases} \quad (2.20)$$

respectively. By Theorem 2.11, we know the existence of κ, \mathbf{v} , and ψ that solve (2.19) and possess the estimate:

$$\begin{aligned} & \|e^{\eta s} \kappa\|_{W_p^1((0, T), W_q^1(\Omega))} + \|e^{\eta s} \partial_s \mathbf{v}\|_{L_p((0, T), L_q(\Omega)^N)} + \|e^{\eta s} \mathbf{v}\|_{L_p((0, T), W_q^2(\Omega)^N)} + \|e^{\eta s} \psi\|_{W_p^1((0, T), W_q^1(\Omega))} \\ & \leq C \left\{ \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \right. \\ & \quad \left. + \|e^{\eta s} (f, \mathbf{H}, \mathbf{k})\|_{L_p((0, T), W_q^{1,0}(\Omega))} + \|e^{\eta s} \mathbf{g}\|_{L_p((0, T), L_p(\Omega))} + \|e^{\eta s} \partial_s \mathbf{k}\|_{L_p((0, T), W_q^{-1}(\mathbb{R}^N))} \right\}. \end{aligned} \quad (2.21)$$

For the sake of simplicity, we set

$$\begin{aligned} \mathbf{J}_{p,q} &= \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \\ & \quad + \|e^{\eta s} (f, \mathbf{H}, \mathbf{k})\|_{L_p((0, T), W_q^{1,0}(\Omega))} + \|e^{\eta s} \mathbf{g}\|_{L_p((0, T), L_p(\Omega))} + \|e^{\eta s} \partial_s \mathbf{k}\|_{L_p((0, T), W_q^{-1}(\mathbb{R}^N))}, \end{aligned}$$

where $\eta = \min(\eta_1, \eta_2)/2$, and η_1 and η_2 are the positive numbers appearing in Corollary 2.10 and Theorem 2.11, respectively. Let $\{T(t)\}_{t \geq 0}$ be the semigroup associated with (2.12) and let $\mathbf{z}(x, s) = \mathbf{v}(x, s) - d(S) \sum_{\ell=1}^M (\mathbf{v}(\cdot, s), \mathbf{p}_\ell)_{\Omega} \mathbf{p}_\ell$. Defining $\tilde{\omega}, \tilde{\mathbf{w}}$, and $\tilde{\varphi}$ by

$$(\tilde{\omega}(\cdot, t), \tilde{\mathbf{w}}(\cdot, t), \tilde{\varphi}(\cdot, t)) = \lambda_0 \int_0^t \mathbf{T}(t-s)(\kappa(\cdot, s), \rho_* \mathbf{z}(\cdot, s), \psi(\cdot, s)) ds \quad (2.22)$$

by the Duhamel principle, we see that $\tilde{\omega}, \tilde{\mathbf{w}}$, and $\tilde{\varphi}$ satisfy the equations

$$\begin{cases} \partial_t \tilde{\omega} + \rho_* \operatorname{div} \tilde{\mathbf{w}} = \lambda_0 \kappa & \text{in } \Omega \times (0, T), \\ \rho_* \partial_t \tilde{\mathbf{w}} - \operatorname{Div} S(\tilde{\mathbf{w}}) + \nabla (P'(\rho_*) \tilde{\omega}) - \beta \operatorname{Div} \tilde{\varphi} = \rho_* \lambda_0 (\mathbf{v} - d(S) \sum_{\ell=1}^M (\mathbf{v}(\cdot, s), \mathbf{p}_\ell)_{\Omega} \mathbf{p}_\ell) & \text{in } \Omega \times (0, T), \\ \partial_t \tilde{\varphi} + \gamma \tilde{\varphi} - \delta D(\tilde{\mathbf{w}}) = \lambda_0 \psi & \text{in } \Omega \times (0, T), \\ \tilde{\mathbf{w}} = 0 & \text{in } S \times (0, T), \\ (S(\tilde{\mathbf{w}}) - P'(\rho_*) \tilde{\omega} \mathbf{I} + \beta \tilde{\varphi}) \cdot \mathbf{n} = 0 & \text{in } \Gamma \times (0, T), \\ (\tilde{\omega}, \tilde{\mathbf{w}}, \tilde{\varphi})|_{t=0} = (0, 0, 0) & \text{in } \Omega. \end{cases} \quad (2.23)$$

Because $(\mathbf{z}(\cdot, s), \mathbf{p}_\ell)_\Omega = 0$ for any $\ell = 1, \dots, M$ and $s \in (0, T)$ when $S = \emptyset$, by Corollary 2.10, we have

$$\|(\tilde{\omega}(\cdot, t), \tilde{\mathbf{w}}(\cdot, t), \tilde{\varphi}(\cdot, t))\|_{W_q^{1,0}(\Omega)} \leq C \int_0^t e^{-\eta_1(t-s)} \|(\kappa(\cdot, s), \mathbf{z}(\cdot, s), \psi(\cdot, s))\|_{W_q^{1,0}(\Omega)} ds.$$

Thus, by Hölder's inequality and the change of the integral order, we have

$$\begin{aligned} & \int_0^T (e^{\eta t} \|(\tilde{\omega}(\cdot, t), \tilde{\mathbf{w}}(\cdot, t), \tilde{\varphi}(\cdot, t))\|_{W_q^{1,0}(\Omega)})^p dt \\ & \leq C \int_0^T \left(\int_0^t e^{-\eta(t-s)} e^{\eta s} \|(\kappa(\cdot, s), \mathbf{z}(\cdot, s), \psi(\cdot, s))\|_{W_q^{1,0}(\Omega)} ds \right)^p dt \\ & = C \int_0^T \int_0^t e^{-\eta(t-s)} (e^{\eta s} \|(\kappa(\cdot, s), \mathbf{z}(\cdot, s), \psi(\cdot, s))\|_{W_q^{1,0}(\Omega)})^p ds \left(\int_0^t e^{-\eta(t-s)} ds \right)^{p/p'} dt \\ & = C \eta^{-p/p'} \int_0^T (e^{\eta s} \|(\kappa(\cdot, s), \mathbf{z}(\cdot, s), \psi(\cdot, s))\|_{W_q^{1,0}(\Omega)})^p ds \int_s^T e^{-\eta(t-s)} dt \\ & = C \eta^{-p} \int_0^T (e^{\eta s} \|(\kappa(\cdot, s), \mathbf{z}(\cdot, s), \psi(\cdot, s))\|_{W_q^{1,0}(\Omega)})^p ds, \end{aligned}$$

which, combined with (2.21), furnishes that

$$\|e^{\eta s}(\tilde{\omega}, \tilde{\mathbf{w}}, \tilde{\varphi})\|_{L_p((0,T), W_q^{1,0}(\Omega))} \leq C \mathbf{J}_{p,q}. \quad (2.24)$$

Because $\tilde{\omega}$, $\tilde{\varphi}$, and $\tilde{\mathbf{w}}$ satisfy the shifted equations

$$\begin{cases} \tilde{\omega}_t + \lambda_0 \tilde{\omega} + \rho_* \operatorname{div} \tilde{\mathbf{w}} = \lambda_0(\tilde{\omega} + \kappa) & \text{in } \Omega \times (0, T), \\ \rho_* (\tilde{\mathbf{w}}_t + \lambda_0 \tilde{\mathbf{w}}) - \operatorname{Div} \mathbf{S}(\tilde{\mathbf{w}}) + \nabla(P'(\rho_*)\tilde{\omega}) - \beta \operatorname{Div} \tilde{\varphi} \\ = \rho_* \lambda_0 (\tilde{\mathbf{w}} + \mathbf{v} - d(S) \sum_{\ell=1}^M (\mathbf{v}(\cdot, s), \mathbf{p}_\ell)_\Omega \mathbf{p}_\ell) & \text{in } \Omega \times (0, T), \\ \partial_t \tilde{\varphi} + \lambda_0 \tilde{\varphi} + \gamma \tilde{\varphi} - \delta \mathbf{D}(\tilde{\mathbf{w}}) = \lambda_0(\tilde{\varphi} + \psi) & \text{in } \Omega \times (0, T), \\ \tilde{\mathbf{w}} = 0 & \text{in } S \times (0, T), \\ (S(\tilde{\mathbf{w}}) - P'(\rho_*)\tilde{\omega} \mathbf{I} + \beta \tilde{\varphi}) \cdot \mathbf{n} = 0 & \text{in } \Gamma \times (0, T), \\ (\tilde{\omega}, \tilde{\mathbf{w}}, \tilde{\varphi})|_{t=0} = (0, 0, 0) & \text{in } \Omega, \end{cases} \quad (2.25)$$

by Theorem 2.11, (2.21), and (2.24), we have

$$\begin{aligned} & \|e^{\eta s} \tilde{\omega}\|_{W_p^1((0,T), W_q^1(\Omega))} + \|e^{\eta s} \partial_s \tilde{\mathbf{w}}\|_{L_p((0,T), L_q(\Omega))} + \|e^{\eta s} \tilde{\mathbf{w}}\|_{L_p((0,T), W_q^2(\Omega))} \\ & + \|e^{\eta s} \tilde{\varphi}\|_{W_p^1((0,T), W_q^1(\Omega))} \leq C \mathbf{J}_{p,q}. \end{aligned} \quad (2.26)$$

When $S \neq \emptyset$, setting $\omega = \tilde{\omega}$, $\varphi = \tilde{\varphi}$, and $\mathbf{w} = \tilde{\mathbf{w}}$, we have Theorem 2.7.

Finally, we consider the case $S = \emptyset$. Let

$$\omega = \tilde{\omega}, \quad \varphi = \tilde{\varphi}, \quad \mathbf{w} = \tilde{\mathbf{w}} + \lambda_0 \rho_* d(S) \sum_{\ell=1}^M \int_0^t (\mathbf{v}(\cdot, s), \mathbf{p}_\ell)_\Omega ds \mathbf{p}_\ell.$$

Because (2.11) holds and \mathbf{p}_ℓ is the first-order polynomial, we have $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v} + \operatorname{div} \tilde{\mathbf{w}}$, $\mathbf{S}(\mathbf{u}) = \mathbf{S}(\mathbf{v}) + \mathbf{S}(\tilde{\mathbf{w}})$, $\mathbf{D}(\mathbf{u}) = \mathbf{D}(\mathbf{v}) + \mathbf{D}(\tilde{\mathbf{w}})$, and $\nabla^2 \mathbf{u} = \nabla^2(\mathbf{v} + \tilde{\mathbf{w}})$. Thus, by (2.22) and (2.25), we see that θ , \mathbf{u} , and τ satisfy Equation 2.1. Moreover, by (2.21) and (2.26), we have

$$\begin{aligned} & \|e^{\eta s} \theta\|_{W_p^1((0,T), W_q^1(\Omega))} + \|e^{\eta s} \partial_s \mathbf{u}\|_{L_p((0,T), L_q(\Omega))} + \|e^{\eta s} \mathbf{D}(\mathbf{u})\|_{L_p((0,T), L_q(\Omega))} \\ & + \|e^{\eta s} \nabla^2 \mathbf{u}\|_{L_p((0,T), L_q(\Omega))} + \|e^{\eta s} \tau\|_{W_p^1((0,T), W_q^1(\Omega))} \leq C \mathbf{J}_{p,q}. \end{aligned} \quad (2.27)$$

Using the first Korn inequality, we have

$$\|\mathbf{u}(\cdot, s)\|_{W_q^1(\Omega)} \leq C \{ \|\mathbf{D}(\mathbf{u}(\cdot, s))\|_{L_q(\Omega)} + \sum_{\ell=1}^M |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega| \},$$

which, combined with (2.27), furnishes that

$$\begin{aligned} \|e^{\eta s} \mathbf{u}(\cdot, s)\|_{L_p((0,T), W_q^1(\Omega))} & \leq C \left\{ \|e^{\eta s} \mathbf{D}(\mathbf{u}(\cdot, s))\|_{L_p((0,T), L_q(\Omega))} + \sum_{\ell=1}^M \left(\int_0^t (e^{\eta s} |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|)^p ds \right)^{1/p} \right\} \\ & \leq C \left\{ \mathbf{J}_{p,q} + \sum_{\ell=1}^M \left(\int_0^t (e^{\eta s} |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|)^p ds \right)^{1/p} \right\} \end{aligned} \quad (2.28)$$

Thus, combining (2.27) and (2.28), we have

$$\begin{aligned} & \|e^{\eta s} \theta\|_{W_p^1((0,T),W_q^1(\Omega))} + \|e^{\eta s} \partial_s \mathbf{u}\|_{L_p((0,T),L_q(\Omega))} + \|e^{\eta s} \mathbf{u}\|_{L_p((0,T),W_q^2(\Omega))} \\ & + \|e^{\eta s} \tau\|_{W_p^1((0,T),W_q^1(\Omega))} \leq C \left\{ \mathbf{J}_{p,q} + \sum_{\ell=1}^M \left(\int_0^t (e^{\eta s} |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds) \right)^{1/p} \right\}. \end{aligned}$$

This completes the proof of Theorem 2.7.

3. A proof of Theorem 2.9

In this section, we prove Theorem 2.9. For this purpose, first, we consider problem (2.2) with $\lambda = 0$, that is

$$\begin{cases} \rho_* \operatorname{div} \mathbf{u} = f & \text{in } \Omega, \\ -\operatorname{Div} \mathbf{S}(\mathbf{u}) + P'(\rho_*) \nabla \theta - \beta \operatorname{Div} \tau = \mathbf{g} & \text{in } \Omega, \\ \gamma \tau - \delta \mathbf{D}(\mathbf{u}) = \mathbf{H} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } S, \\ (\mathbf{S}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I} + \beta \tau) \mathbf{n} = \mathbf{k} & \text{on } \Gamma. \end{cases} \quad (3.1)$$

We start with

Lemma 3.1

Let $1 < q < \infty$, $N < r < \infty$, and $\lambda_1 > 0$. Assume that $\max(q, q') \leq r$. Let ℓ_b be the number defined in Theorem 1.1. Let Ω be a bounded domain in \mathbb{R}^N , whose boundaries S and Γ are both $W_r^{\ell_b-1/r}$ compact hyper-surfaces. Then, for any $(f, \mathbf{g}, \mathbf{H}) \in W_q^{1,0}(\Omega)$ and $\mathbf{k} \in W_q^1(\Omega)^N$ satisfying the condition:

$$(\mathbf{g}, \mathbf{p}_\ell)_\Omega + (\mathbf{k}, \mathbf{p}_\ell)_\Gamma = 0 \quad (\ell = 1, \dots, M) \quad (3.2)$$

when $S = \emptyset$, problem (3.1) admits unique solutions $\theta \in W_q^1(\Omega)$ and $\mathbf{u} \in \dot{W}_q^2(\Omega)$ possessing the estimate:

$$\|(\theta, \mathbf{u}, \tau)\|_{W_q^{1,2}(\Omega)} \leq C(\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} + \|\mathbf{k}\|_{W_q^1(\Omega)}). \quad (3.3)$$

Remark 3.2

Recall that $\dot{W}_q^2(\Omega)^N$ is the set of all $\mathbf{u} \in W_q^2(\Omega)^N$, which satisfies (2.10).

Proof

Setting $\tau = \gamma^{-1}(\delta \mathbf{D}(\mathbf{u}) + \mathbf{H})$ in (3.1) and inserting the relation $\operatorname{div} \mathbf{u} = \rho_*^{-1} f$ in the second equation and boundary condition, we have

$$\begin{cases} \rho_* \operatorname{div} \mathbf{u} = f & \text{in } \Omega, \\ -\operatorname{Div} (\mu' \mathbf{D}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I}) = \mathbf{g}' & \text{in } \Omega, \\ (\mu' \mathbf{D}(\mathbf{u}) - P'(\rho_*) \theta \mathbf{I}) \mathbf{n} = \mathbf{k}' & \text{on } \Gamma. \end{cases} \quad (3.4)$$

with $\mu' = \mu + \gamma^{-1} \delta \beta$, $\mathbf{g}' = \mathbf{g} + \operatorname{Div}((v - \mu) \rho_*^{-1} f \mathbf{I} + \gamma^{-1} \beta \mathbf{H})$ and $\mathbf{k}' = \mathbf{k} - ((v - \mu) \rho_*^{-1} f \mathbf{I} + \gamma^{-1} \beta \mathbf{H}) \mathbf{n}$. In the sequel, we use the estimate:

$$\|h \mathbf{n}\|_{W_q^i(\Omega)} \leq C \|h\|_{W_q^i(\Omega)} \quad (i = 0, 1), \quad (3.5)$$

which was proved for example in Maryani [15]. By (3.5),

$$\|\mathbf{g}'\|_{L_q(\Omega)} + \|\mathbf{k}'\|_{W_q^1(\Omega)} \leq C(\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)}).$$

Moreover, by (3.2) and the divergence theorem of Gauß, we have

$$(\mathbf{g}', \mathbf{p}_\ell)_\Omega + (\mathbf{k}', \mathbf{p}_\ell)_\Gamma = 0 \quad (\ell = 1, \dots, M). \quad (3.6)$$

To quote a result due to Shibata [31], we introduce the space $W_{q',\Gamma}^1(\Omega)$ defined by

$$W_{q',\Gamma}^1(\Omega) = \{\varphi \in W_{q'}^1(\Omega) \mid \varphi|_\Gamma = 0\}.$$

For any $\varphi \in W_{q'}^1(\Omega)$, it holds that the Poincarés inequality:

$$\|\varphi\|_{L_{q'}(\Omega)} \leq C \|\nabla \varphi\|_{L_{q'}(\Omega)} \quad \text{for any } \varphi \in W_{q',\Gamma}^1(\Omega). \quad (3.7)$$

Then, by (3.7), we have

$$|(f, \varphi)_\Omega| \leq \|f\|_{L_q(\Omega)} \|\varphi\|_{L_{q'}(\Omega)} \leq C \|f\|_{L_q(\Omega)} \|\nabla \varphi\|_{L_{q'}(\Omega)},$$

which implies that $f \in W_q^{1,1}(\Omega)$ in terms of the terminology defined in Shibata [31][§]. By the Hahn–Banach theorem, there exists an $\mathbf{F} \in L_q(\Omega)^N$ such that $f = \operatorname{div} \mathbf{F}$ and $\|\mathbf{F}\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}$.

Instead of (3.4), first, we consider the equations:

$$\begin{cases} \rho_* \operatorname{div} \mathbf{v} = f & \text{in } \Omega, \\ \lambda_0 \mathbf{v} - \operatorname{Div}(\mu' \mathbf{D}(\mathbf{v}) - P'(\rho_*) \mathbf{p} \mathbf{l}) = \mathbf{g}' & \text{in } \Omega, \\ (\mu' \mathbf{D}(\mathbf{u}) - P'(\rho_*) \mathbf{p} \mathbf{l}) \mathbf{n} = \mathbf{k}' & \text{on } \Gamma. \end{cases} \quad (3.8)$$

According to [31], there exists a large $\lambda_0 > 0$ such that problem (3.8) admits unique solutions $\mathbf{v} \in W_q^2(\Omega)^N$ and $\mathbf{p} \in W_q^1(\Omega)$ possessing the estimate:

$$\|\mathbf{v}\|_{W_q^2(\Omega)} + \|\mathbf{p}\|_{W_q^1(\Omega)} \leq C(\|f\|_{W_q^1(\Omega)} + \|\mathbf{g}'\|_{L_q(\Omega)} + \|\mathbf{k}'\|_{W_q^1(\Omega)}).$$

Moreover, by the divergence theorem of Gauß, we have

$$\lambda_0(\mathbf{v}, \mathbf{p}_\ell) = (\mathbf{g}', \mathbf{p}_\ell)_\Omega + (\mathbf{k}', \mathbf{p}_\ell)_\Gamma - \frac{\mu'}{2}(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{p}_\ell))_\Omega - P'(\rho_*)(\mathbf{p}, \operatorname{div} \mathbf{p}_\ell)_\Omega.$$

By (2.11) and (3.6), we have $(\mathbf{v}, \mathbf{p}_\ell)_\Omega = 0$ ($\ell = 1, \dots, M$), so that $\mathbf{v} \in \dot{W}_q^2(\Omega)^N$.

Setting $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\theta = \mathbf{p} + \mathbf{q}$ in (3.4), we have

$$\begin{cases} \rho_* \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega, \\ -\operatorname{Div}(\mu' \mathbf{D}(\mathbf{w}) - P'(\rho_*) \mathbf{q} \mathbf{l}) = \lambda_0 \mathbf{v} & \text{in } \Omega, \\ (\mu' \mathbf{D}(\mathbf{w}) - P'(\rho_*) \mathbf{q} \mathbf{l}) \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \quad (3.9)$$

The problem

$$\begin{cases} \rho_* \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega, \\ \lambda \mathbf{v} - \operatorname{Div}(\mu' \mathbf{D}(\mathbf{w}) - P'(\rho_*) \mathbf{q} \mathbf{l}) = \lambda_0 \mathbf{v} & \text{in } \Omega, \\ (\mu' \mathbf{D}(\mathbf{w}) - P'(\rho_*) \mathbf{q} \mathbf{l}) \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \quad (3.10)$$

is uniquely solvable according to a result due to Shibata [31], and the inverse operator concerning problem (3.10) is compact, so that by the Riesz–Schauder theorem, the uniqueness implies the existence in problem (3.9). This fact is called by the homotopic argument in the sequel. Thus, we examine the uniqueness of problem (3.9). Let $\mathbf{w} \in \dot{W}_q^2(\Omega)$ and $\mathbf{q} \in W_q^1(\Omega)$ satisfy the homogeneous equation:

$$\begin{cases} \rho_* \operatorname{div} \mathbf{w} = 0 & \text{in } \Omega, \\ -\operatorname{Div}(\mu' \mathbf{D}(\mathbf{w}) - P'(\rho_*) \mathbf{q} \mathbf{l}) = 0 & \text{in } \Omega, \\ (\mu' \mathbf{D}(\mathbf{w}) - P'(\rho_*) \mathbf{q} \mathbf{l}) \mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \quad (3.11)$$

First, we consider the case where $2 \leq q < \infty$. Then, $\mathbf{w} \in \dot{W}_2^2(\Omega)$ and $\mathbf{q} \in W_2^1(\Omega)$, so that multiplying the second equation of (3.11) by \mathbf{w} , integrating the resultant formulas on Ω , and using the divergence theorem of Gauß, we have

$$\frac{\mu'}{2} \|\mathbf{D}(\mathbf{w})\|_{L_2(\Omega)}^2 = 0$$

because $\operatorname{div} \mathbf{w} = 0$. Thus, $\mathbf{D}(\mathbf{w}) = 0$, namely, $\mathbf{w} \in \mathcal{R}_d$. On the other hand, \mathbf{w} satisfies (2.10), so that $\mathbf{w} = 0$. Moreover, by the boundary condition, we have $\mathbf{q} = 0$ too. Thus, we have the uniqueness, so that the Riesz–Schauder theorem implies the unique existence of solutions to problem (3.9). When $1 < q < 2$, the uniqueness follows from the existence of the dual problem with exponent $q' > 2$. Then, we also have the uniqueness when $1 < q < 2$, and therefore the Riesz–Schauder theorem implies the unique existence of solutions to problem (3.9) when $1 < q < 2$. Summing up, we have proved the unique existence of solutions $\theta \in W_q^1(\Omega)$, $\mathbf{u} \in \dot{W}_q^2(\Omega)^N$, and $\tau \in W_q^1(\Omega)^{N \times N}$ for problem (3.1). The estimate (3.3) follows from the Banach closed graph theorem, which completes the proof. \square

In the sequel, we prove Theorem 2.9. In view of Lemma 3.1 by the small perturbation argument, there exists a small $\lambda_0 > 0$ such that problem (2.2) can be solved with $\lambda \in \mathbb{C}$ and $|\lambda| \leq \lambda_0$. Namely, Theorem 2.9 holds for $\lambda \in \mathbb{C}$ with $|\lambda| \leq \lambda_0$. Thus, in the sequel, we consider the case where $\operatorname{Re} \lambda \geq 0$ and $\lambda_0 \leq |\lambda| \leq \lambda_1$. In this case, setting $\theta = \lambda^{-1}(f - \rho_* \operatorname{div} \mathbf{u})$ and $\tau = (\lambda + \gamma)^{-1}(\delta \mathbf{D}(\mathbf{u}) + \mathbf{H})$ in (2.2), we have a generalized Lamé system:

$$\rho_* \lambda \mathbf{u} - \operatorname{Div} \mathbf{S}_\lambda(\mathbf{u}) = \mathbf{g}' \text{ in } \Omega, \quad \mathbf{u} = 0 \text{ on } S, \quad \mathbf{S}_\lambda(\mathbf{u}) \mathbf{n} = \mathbf{k}' \text{ on } \Gamma, \quad (3.12)$$

where we have set

$$\begin{aligned} \mathbf{S}_\lambda(\mathbf{u}) &= (\mu + \beta(\lambda + \gamma)^{-1} \delta) \mathbf{D}(\mathbf{u}) + ((\nu - \mu) + P'(\rho_*) \rho_* \lambda^{-1}) \operatorname{div} \mathbf{u}, \\ \mathbf{g}' &= \mathbf{g} - (P'(\rho_*) \lambda^{-1} \nabla f - \beta(\lambda + \gamma)^{-1} \operatorname{Div} \mathbf{H}), \\ \mathbf{k}' &= \mathbf{k} + (P'(\rho_*) \lambda^{-1} \mathbf{f} \mathbf{l} - \beta(\lambda + \gamma)^{-1} \mathbf{H}) \mathbf{n}. \end{aligned}$$

[§] The space $W_q^{1,1}(\Omega)$ was defined in [31], but we just quote a result because of Shibata [31], so that we do not state its definition.

Because $\lambda_0 \leq |\lambda| \leq \lambda_1$, by (3.5)

$$\|\mathbf{g}'\|_{L_q(\Omega)} + \|\mathbf{k}'\|_{W_q^1(\Omega)} \leq C_{\lambda_0, \lambda_1} (\|(f, \mathbf{g}, \mathbf{H})\|_{W_q^{1,0}(\Omega)} + \|\mathbf{k}\|_{W_q^1(\Omega)}).$$

To solve (3.12), first, for fixed λ , we consider the equations:

$$\rho_* \kappa \mathbf{u} - \text{Div } \mathbf{S}_\lambda(\mathbf{u}) = \mathbf{g}' \text{ in } \Omega, \quad \mathbf{u} = 0 \text{ on } S, \quad \mathbf{S}_\lambda(\mathbf{u})\mathbf{n} = \mathbf{k}' \text{ on } \Gamma, \quad (3.13)$$

with new resolvent parameter $\kappa \in \mathbb{R}$. Note that if (\mathbf{g}, \mathbf{k}) satisfies (3.2), then $(\mathbf{g}', \mathbf{k}')$ also satisfies (3.2). Employing the same argumentation as in Shibata and Tanaka [32] or Enomoto, von Below and Shibata [24], we see that there exists a large $\kappa_0 > 0$ depending on λ such that for any $\kappa \geq \kappa_0$ and $(\mathbf{g}', \mathbf{k}') \in L_q(\Omega)^N \times W_q^1(\Omega)^N$ satisfying (3.2), problem (3.13) admits a unique solution $\mathbf{u} \in \dot{W}_q^2(\Omega)^N$. Because the solution operator of problem (3.13) with $\kappa = \kappa_0$ is compact, by the Riesz-Schauder theory, we see that the uniqueness implies the existence in problem (3.12). Thus, we examine the uniqueness. Let $\mathbf{u} \in \dot{W}_q^2(\Omega)^N$ be a solution of the homogeneous equation:

$$\rho_* \lambda \mathbf{u} - \text{Div } \mathbf{S}_\lambda(\mathbf{u}) = 0 \text{ in } \Omega, \quad \mathbf{u} = 0 \text{ on } S, \quad \mathbf{S}_\lambda(\mathbf{u})\mathbf{n} = 0 \text{ on } \Gamma, \quad (3.14)$$

First, we consider the case $2 \leq q < \infty$. In this case, $\mathbf{u} \in \dot{W}_2^2(\Omega)^N$. Thus, multiplying the first Equation 3.14 by \mathbf{u} and using the divergence theorem of Gauß, we have

$$0 = \rho_* \lambda \|\mathbf{u}\|_{L_2(\Omega)}^2 + \frac{1}{2}(\mu + \beta(\lambda + \gamma)^{-1}\delta) \|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}^2 + ((v - \mu) + P'(\rho_*)\rho_* \lambda^{-1}) \|\text{div } \mathbf{u}\|_{L_2(\Omega)}^2. \quad (3.15)$$

When $\text{Re } \lambda \geq 0$, $\text{Re } \rho_* \lambda^{-1} \geq 0$ and $\text{Re } \beta(\lambda + \gamma)^{-1}\delta \geq 0$, so that taking the real part of (3.15), we have

$$0 \geq \rho_* \text{Re } \lambda \|\mathbf{u}\|_{L_2(\Omega)}^2 + \frac{\mu}{2} \|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}^2 + (v - \mu) \|\text{div } \mathbf{u}\|_{L_2(\Omega)}^2. \quad (3.16)$$

Because $\|\text{div } \mathbf{u}\|_{L_2(\Omega)}^2 \leq (N/4) \|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}^2$, by (3.16), we have

$$0 \geq \left(v - \frac{N-2}{N} \mu \right) \|\text{div } \mathbf{u}\|_{L_2(\Omega)}^2$$

provided that $\text{Re } \lambda \geq 0$. Because we assume that $v - \frac{N-2}{N} \mu > 0$, we have $\text{div } \mathbf{u} = 0$, so that by (3.16) and the assumption that $\mu > 0$, we have $\mathbf{D}(\mathbf{u}) = 0$, provided that $\text{Re } \lambda \geq 0$. When $S \neq \emptyset$, we have $\mathbf{u}|_S = 0$, so that the first Korn inequality $\|\nabla \mathbf{u}\|_{L_2(\Omega)} \leq C \|\mathbf{D}(\mathbf{u})\|_{L_2(\Omega)}$ does hold. Therefore, $\nabla \mathbf{u} = 0$, which implies that \mathbf{u} is constant. But, $\mathbf{u}|_S = 0$, so that finally we arrive at $\mathbf{u} = 0$. On the other hand, when $S = \emptyset$, \mathbf{u} satisfies (2.10), so that $\mathbf{u} = 0$ too. Therefore, we have the uniqueness, which implies the unique existence of solutions to problem (3.12) for each λ with $\lambda_0 \leq |\lambda| \leq \lambda_1$ when $2 \leq q < \infty$. When $1 < q < 2$, the uniqueness follows from the existence for the dual problem, so that in this case, we also have the unique existence of solutions. If we know the unique existence of solutions to (3.12) for one λ_2 , by the small perturbation argument, there exists a small number δ depending on λ_2 such that the unique existence of solutions to (3.12) holds for $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_2| \leq \delta$. Because the set $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda \geq 0, \lambda_0 \leq |\lambda| \leq \lambda_1\}$ is compact, we have the unique existence theorem holds for any $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda \geq 0, \lambda_0 \leq |\lambda| \leq \lambda_1\}$ with uniform constant C in the estimate (3.3). This completes the proof of Theorem 2.9.

4. A proof of Theorem 1.3

To prove Theorem 1.3, we start with

Lemma 4.1

Let $1 < p, q < \infty$, let T be any positive number, and let Ω be a bounded domain in \mathbb{R}^N , whose boundary Γ is a $W_r^{2-1/r}$ compact hyper-surface with $N < r < \infty$. Then, the following two assertions hold:

1. We have

$$\sup_{t \in (0, T)} \|u(t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C \{ \|u(\cdot, 0)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \mathbf{I}_u(T) \}$$

for any $u \in L_p((0, T), W_q^2(\Omega)) \cap W_p^1((0, T), L_q(\Omega))$ with some constant C independent of T . Here, we have set

$$\mathbf{I}_u(T) = \|\partial_t u\|_{L_p((0, T), L_q(\Omega))} + \|u\|_{L_p((0, T), W_q^2(\Omega))}.$$

2. Assume that $\max(q, q') \leq r$. Then, we have

$$\begin{aligned} \|\nabla u\|_{W_q^{-1}(\Omega)} &\leq C \|u\|_{L_q(\Omega)} && \text{for any } u \in L_q(\Omega), \\ \|uv\|_{W_q^{-1}(\Omega)} &\leq C \|u\|_{W_q^{-1}(\Omega)} \|v\|_{W_q^1(\Omega)} && \text{for any } u \in W_q^{-1}(\Omega), v \in W_q^1(\Omega), \\ \|uv\|_{W_q^{-1}(\Omega)} &\leq C \|u\|_{L_q(\Omega)} \|v\|_{L_q(\Omega)} && \text{for any } u, v \in L_q(\Omega). \end{aligned} \quad (4.1)$$

Proof

Lemma has been proved in [15] (cf. also in [31]), so that we may omit the proof. \square

From now on, we prove Theorem 1.3. Let ϵ be a small positive number, and we assume that initial data $(\theta_0, \mathbf{v}_0, \tau_0) \in \mathcal{D}_{q,p}(\Omega)$ satisfy the conditions:

$$\begin{aligned} \frac{2}{3}\rho_* < \rho_* + \theta_0 < \frac{4}{3}\rho_*, \\ \|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \leq \epsilon \end{aligned} \quad (4.2)$$

and the orthogonal condition (1.18). Because we choose an ϵ small enough eventually, we may assume that $0 < \epsilon \leq 1$. Thus, by Theorem 1.1, there exists a $T_0 > 0$ such that problem (1.10) admits a unique solution with $T = T_0$. Let T be a positive number, and we assume that problem (1.10) admits a solution $(\theta, \mathbf{u}, \omega)$ with

$$\theta \in W_p^1((0, T), W_q^1(\Omega)), \mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, \infty), L_q(\Omega)^N), \omega \in W_p^1((0, T), W_q^1(\Omega)^{N \times N})$$

satisfying the condition:

$$\frac{1}{3}\rho_* < \rho_* + \theta(x, t) < \frac{5}{3}\rho_* \quad \text{for any } (x, t) \in \Omega \times (0, T), \quad \sup_{0 < t < T} \left\| \int_0^t \nabla \mathbf{u}(\cdot, s) ds \right\|_{L_\infty(\Omega)} \leq \sigma, \quad (4.3)$$

where σ is the positive number appearing in (1.7). We may assume that $0 < \sigma \leq 1$ and $T \geq T_0$. Let

$$\mathbf{I}(t) = \|e^{\eta s} \theta\|_{W_p^1((0, t), W_q^1(\Omega))} + \|e^{\eta s} \partial_s \mathbf{u}\|_{L_p((0, t), L_q(\Omega))} + \|e^{\eta s} \mathbf{u}\|_{L_p((0, t), W_q^2(\Omega))} + \|e^{\eta s} \omega\|_{W_p^1((0, t), W_q^1(\Omega))}$$

with some positive constant η for which Theorem 2.7 holds. Our main task is to prove

$$\mathbf{I}(t) \leq M_1(\epsilon + \mathbf{I}(t)^2) \quad (4.4)$$

with some constant M_1 independent of ϵ and T . To prove (4.4), we start with

$$\begin{aligned} \|\theta(\cdot, t)\|_{W_q^1(\Omega)} &\leq C(\|\theta_0\|_{W_q^1(\Omega)} + \mathbf{I}(t)), \\ \|\mathbf{u}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} &\leq C(\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \mathbf{I}(t)), \\ \|\omega(\cdot, t)\|_{W_q^1(\Omega)} &\leq C(\|\tau_0\|_{W_q^1(\Omega)} + \mathbf{I}(t)). \end{aligned} \quad (4.5)$$

In fact, writing $\theta(x, t) = \theta_0 + \int_0^t \partial_s \theta(\cdot, s) ds$ and $\omega(x, t) = \tau_0 + \int_0^t \partial_s \omega(\cdot, s) ds$, we have the first and third inequality in (4.5). The second inequality in (4.5) follows from Lemma 4.1 (1). Hereinafter, the letter C stands for generic constants independent of T and ϵ . Its value may differ even in a single chain of inequalities. By Hölder's inequality, we have

$$\int_0^t \|\mathbf{u}(\cdot, s)\|_{W_q^2(\Omega)} ds \leq C \left(\int_0^t e^{-p's} ds \right)^{1/p'} \left(\int_0^t (e^{\gamma s} \|\mathbf{u}(\cdot, s)\|_{W_q^2(\Omega)})^p ds \right)^{1/p} \leq C \mathbf{I}(t). \quad (4.6)$$

To estimate the products, we use the Sobolev embedding theorem:

$$\prod_{j=1}^m \|f_j\|_{W_q^1(\Omega)} \leq C \prod_{j=1}^m \|f_j\|_{W_q^1(\Omega)}, \quad \|f\|_{L_\infty(\Omega)} \leq C \|f\|_{W_q^1(\Omega)} \quad (4.7)$$

because $N < q < \infty$. Because $2 < p < \infty$, we have $B_{q,p}^{2(1-1/p)}(\Omega) \subset W_q^1(\Omega)$, that is,

$$\|f\|_{W_q^1(\Omega)} \leq C \|f\|_{B_{q,p}^{2(1-1/p)}(\Omega)}. \quad (4.8)$$

Recall the definition of nonlinear terms $f(\theta, \mathbf{u}, \omega)$, $\mathbf{g}(\theta, \mathbf{u}, \omega)$, $\mathbf{h}(\theta, \mathbf{u}, \omega)$, and $\mathbf{L}(\theta, \mathbf{u}, \omega)$. Using (3.5), (4.3), (4.5), (4.6), (4.7), and (4.8) and noting that $V_{\text{div}}(0) = 0$, $V_D(0) = 0$, $V_{\text{Div}}(0) = 0$, and $V_0(0) = 0$, we have

$$\|e^{\gamma s} (f(\theta, \mathbf{u}, \omega), \mathbf{h}(\theta, \mathbf{u}, \omega), \mathbf{L}(\theta, \mathbf{u}, \omega))\|_{L_p((0, t), W_q^1(\Omega))} + \|e^{\gamma s} \mathbf{g}(\theta, \mathbf{u}, \omega)\|_{L_p((0, t), L_q(\Omega))} \leq C(\epsilon + \mathbf{I}(t)^2). \quad (4.9)$$

By (3.5), (4.1), (4.3), (4.5), (4.6), (4.7), and (4.8) and noting that $V_{\text{div}}(0) = 0$, $V_D(0) = 0$, $V_{\text{Div}}(0) = 0$, and $V_0(0) = 0$, we also have

$$\|e^{\gamma s} \partial_s \mathbf{h}(\theta, \mathbf{u}, \omega)\|_{L_p((0, t), W_q^{-1}(\mathbb{R}^N))} \leq C(\epsilon + \mathbf{I}(t)^2). \quad (4.10)$$

To obtain (4.9) and (4.10), we used the fact that $(\epsilon + \mathbf{I}(t))\mathbf{I}(t) \leq (1/2)\epsilon^2 + (3/2)\mathbf{I}(t)^2 \leq 2(\epsilon + \mathbf{I}(t)^2)$, because of $0 < \epsilon \leq 1$.

Applying Theorem 2.7 to problem (1.10) and using (4.9) and (4.10), we have

$$\mathbf{I}(t) \leq C\{\epsilon + \mathbf{I}(t)^2 + d(S) \sum_{\ell=1}^M \left(\int_0^t (e^{\gamma s} |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)|_\Omega)^p ds \right)^{1/p}\}. \quad (4.11)$$

Now, we consider the case where $S = \emptyset$, namely, $d(S) = 1$. According to the argumentation due to G. Ströhmer [27], the Lagrange transform $x = \mathbf{X}_u(\xi, t) = \xi + \int_0^t \mathbf{u}(\xi, s) ds$ is a bijection from Ω onto $\Omega_t = \{x = \mathbf{X}_u(\xi, t) \mid \xi \in \Omega\}$ and from Γ onto $\Gamma_t = \{x = \mathbf{X}_u(\xi, t) \mid \xi \in S\}$, so that denoting the inverse map by $\mathbf{Y}(x, t)$, by (1.9), we see that $\rho(x, t) = \rho_* + \theta(\mathbf{Y}(x, t), t)$, $\mathbf{v}(x, t) = \mathbf{u}(\mathbf{Y}(x, t), t)$, and $\tau(x, t) = \omega(\mathbf{Y}(x, t), t)$ satisfy Equation 1.1. Because we assume that $\tau_0 \in \text{Sym}(\mathbb{R}^N)$, we know that $\tau \in \text{Sym}(\mathbb{R}^N)$, too. Let J be the determinant of the Jacobi matrix of the transformation: $x = \mathbf{X}_u(\xi, t)$, and then noting that $\rho(\xi + \int_0^t \mathbf{u}(\xi, s) ds, t) = \rho_* + \theta(\xi, t)$ and $\mathbf{v}(\xi + \int_0^t \mathbf{u}(\xi, s) ds, t) = \mathbf{u}(\xi, t)$, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega_t} (\rho(\cdot, t) \mathbf{v}(\cdot, t), \mathbf{p}_\ell) dx \\ &= \frac{d}{dt} \left[\int_{\Omega} (\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t) \cdot \mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds) J(\xi, t) d\xi \right] \\ &= \int_{\Omega} \partial_t [(\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t)] \cdot \mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds) J(\xi, t) d\xi \\ &\quad + \int_{\Omega} (\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t) \cdot \partial_t \left[\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds) \right] J(\xi, t) d\xi \\ &\quad + \int_{\Omega} (\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t) \cdot \mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds) \partial_t J(\xi, t) d\xi. \end{aligned}$$

Because $\partial_t J(\xi, t) = (\text{div } \mathbf{v}(x, t)) J(\xi, t)$, by (1.1), we have

$$\begin{aligned} & \partial_t ((\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t)) J(\xi, t) + (\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t) \partial_t J(\xi, t) \\ &= (\partial_t (\rho \mathbf{v}) + \mathbf{v} \cdot \nabla (\rho \mathbf{v}) + \rho \text{div } \mathbf{v}) J(\xi, t) \\ &= [\rho (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) + (\partial_t \rho + \text{div}(\rho \mathbf{v})) \mathbf{v}] J(\xi, t) \\ &= (\text{Div } \mathbf{T}(\mathbf{v}, \rho) + \beta \text{Div } \tau) J(\xi, t). \end{aligned}$$

Moreover, representing $\mathbf{p}_\ell(x) = \left(\sum_{j=1}^N a_{ij} x_j, \dots, \sum_{j=1}^N a_{Nj} x_j \right) + \mathbf{b}$ with $a_{ij} + a_{ji} = 0$, we have

$$\mathbf{u}(\xi, t) \cdot \partial_t (\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds)) = \sum_{i,j=1}^N a_{ij} u_i(\xi, t) u_j(\xi, t) = \frac{1}{2} \sum_{i,j=1}^N (a_{ij} + a_{ji}) u_i(\xi, t) u_j(\xi, t) = 0.$$

Summing up these two facts and using the symmetry of τ and (2.11), we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega_t} (\rho(\cdot, t) \mathbf{v}(\cdot, t), \mathbf{p}_\ell) dx = (\text{Div } \mathbf{T}(\mathbf{v}, \rho) + \beta \text{Div } \tau, \mathbf{p}_\ell)_{\Omega_t} \\ &= -\frac{\mu}{2} (\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{p}_\ell))_{\Omega_t} - (v - \mu)(\text{div } \mathbf{v}, \text{div } \mathbf{p}_\ell)_{\Omega_t} + (P(\rho), \text{div } \mathbf{p}_\ell)_{\Omega_t} - \frac{\beta}{2} (\tau, \mathbf{D}(\mathbf{p}_\ell))_{\Omega_t} = 0. \end{aligned}$$

Thus,

$$\int_{\Omega} (\rho_* + \theta(\xi, t)) \mathbf{u}(\xi, t) \mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds) J(\xi, t) d\xi = ((\rho_* + \theta_0) \mathbf{v}_0, \mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1, \dots, M) \quad (4.12)$$

for any $t \in (0, T)$. Because $J(\xi, t) = \det(\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla \mathbf{u}(\xi, s) ds))$ and $\mathbf{V}_0(0) = 0$, we may write $J(\xi, t)$ in the form:

$$J(\xi, t) = 1 + v_0 \left(\int_0^t \nabla \mathbf{u}(\xi, s) ds \right)$$

where $v_0 = v_0(\mathbf{K})$ is a C^∞ function with respect to \mathbf{K} defined on $|\mathbf{K}| \leq \sigma$ with $v_0(0) = 0$. Moreover, we write

$$\mathbf{p}_\ell(\xi + \int_0^t \mathbf{u}(\xi, s) ds) = \mathbf{p}_\ell(\xi) + A_\ell \int_0^t \mathbf{u}(\xi, s) ds$$

with some constant matrix A_ℓ . And then, by (4.12), we have

$$\begin{aligned} (\mathbf{u}(\cdot, t), \mathbf{p}_\ell)_\Omega &= -\rho_*^{-1} \left(\rho_* \left(\mathbf{u}(\cdot, t), \mathbf{p}_\ell v_0 \left(\int_0^t \nabla \mathbf{u}(\xi, s) ds \right) \right) \right)_\Omega \\ &+ \rho_* (\mathbf{u}(\cdot, t), A_\ell \int_0^t \mathbf{u}(\cdot, s) ds J(\cdot, t))_\Omega + \int_\Omega \theta(\xi, t) \mathbf{u}(\xi, t) \mathbf{p}_\ell (\xi + \int_0^t \mathbf{u}(\xi, s) ds) J(\xi, t) d\xi. \end{aligned} \quad (4.13)$$

Thus, using (4.3) and (4.13), we have

$$|(\mathbf{u}(\cdot, t), \mathbf{p}_\ell)_\Omega| \leq C(\|\theta_0\|_{W_q^1(\Omega)} + \mathbf{I}(t)) \|\mathbf{u}(\cdot, t)\|_{L_q(\Omega)}, \quad (4.14)$$

which furnishes that

$$\sum_{\ell=1}^M \left(\int_0^t (e^{\gamma s} |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds) \right)^{1/p} \leq C(\epsilon + \mathbf{I}(t)^2). \quad (4.15)$$

Combining (4.11) and (4.15), we have (4.4).

Finally, using (4.4), we show that solutions can be prolonged to any time interval beyond $(0, T)$. Let $r_\pm(\epsilon) = (2M_1)^{-1} \pm \sqrt{(2M_1)^{-2} - \epsilon}$ be the two roots of the quadratic equation: $M_1(x^2 + \epsilon) - x = 0$. If $0 < \epsilon < (2M_1)^{-2}$, then $0 < r_-(\epsilon) < r_+(\epsilon)$ and $r_-(\epsilon) = M_1\epsilon + O(\epsilon^2)$ as $\epsilon \rightarrow 0$. Because $\mathbf{I}(t) \rightarrow 0$ as $t \rightarrow 0$ and $\mathbf{I}(t)$ is continuous with respect to t as long as solutions exist, there exists an $\epsilon_0 \in (0, 1)$ such that

$$\mathbf{I}(t) \leq r_-(\epsilon) \leq 2M_1\epsilon \quad (4.16)$$

for any $t \in (0, T)$ and $\epsilon \in (0, \epsilon_0)$. By (4.5),

$$\|\theta(\cdot, T)\|_{W_q^1(\Omega)} + \|\mathbf{u}(\cdot, T)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|\omega(\cdot, T)\|_{W_q^1(\Omega)} \leq M_2\epsilon \leq M_2 \quad (4.17)$$

with some constant M_2 independent of ϵ . By (4.7), $\|\theta(\cdot, T)\|_{L_\infty(\Omega)} \leq C\|\theta(\cdot, T)\|_{W_q^1(\Omega)} \leq CM_2\epsilon$, so that choosing ϵ so small that $CM_2\epsilon < (1/3)\rho_*$, we have

$$\frac{2}{3}\rho_* < \rho_* + \theta(x, T) < \frac{4}{3}\rho_*. \quad (4.18)$$

We consider the nonlinear equations:

$$\left\{ \begin{array}{ll} \partial_t \bar{\theta} + \rho_* \operatorname{div} \bar{\mathbf{u}} = \tilde{f}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega}) & \text{in } \Omega \times (T, T + T_1) \\ \rho_* \partial_t \bar{\mathbf{u}} - \operatorname{Div} \mathbf{S}(\bar{\mathbf{u}}) + P'(\rho_*) \nabla \bar{\theta} - \beta \operatorname{Div} \bar{\omega} = \tilde{\mathbf{g}}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega}) & \text{in } \Omega \times (T, T + T_1) \\ \partial_t \bar{\omega} + \gamma \bar{\omega} - \delta \mathbf{D}(\bar{\mathbf{u}}) = \tilde{\mathbf{L}}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega}) & \text{in } \Omega \times (T, T + T_1) \\ (S(\bar{\mathbf{u}}) - P'(\rho_*) \bar{\theta} \mathbf{I} + \beta \bar{\omega}) \mathbf{n} = \tilde{\mathbf{h}}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega}) & \text{on } \Gamma_1 \times (T, T + T_1) \\ \bar{\mathbf{u}} = 0 & \text{on } S \times (T, T + T_1) \\ (\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega})|_{t=T} = (\theta(\cdot, T), \mathbf{u}(\cdot, T), \omega(\cdot, T)) & \text{in } \Omega \end{array} \right. \quad (4.19)$$

which is the corresponding equations to main problem for time interval $(T, T + T_1)$. Here, $\tilde{f}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega})$, $\tilde{\mathbf{g}}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega})$, $\tilde{\mathbf{L}}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega})$, and $\tilde{\mathbf{h}}(\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega})$ are nonlinear functions defined by replacing $\theta, \mathbf{u}, \omega$, and $\int_0^t \nabla \mathbf{u} ds$ by $\bar{\theta}, \bar{\mathbf{u}}, \bar{\omega}$, and $\int_0^T \nabla \mathbf{u} ds + \int_T^t \nabla \bar{\mathbf{u}} ds$ in (1.12), respectively. Because $\int_0^T \|\nabla \mathbf{u}(\cdot, s)\|_{L_\infty} ds \leq C\epsilon$ as follows from (4.7) and (4.16), employing the same argumentation as in the proof of the local well-posedness for problem (1.10) because of Maryani [15] or the local well-posedness for the compressible barotropic viscous fluid flow because of Enomoto, von Below, and Shibata [24], we can choose positive numbers ϵ and T_1 so small that problem (4.19) admits unique solutions $\bar{\theta}, \bar{\mathbf{u}}$, and $\bar{\omega}$ with

$$\begin{aligned} \bar{\theta} &\in W_p^1((T, T + T_1), W_q^1(\Omega)), \quad \bar{\mathbf{u}} \in L_p((T, T + T_1), W_q^2(\Omega)^N) \cap W_p^1((T, T + T_1), L_q(\Omega)^N), \\ \bar{\omega} &\in W_p^1((T, T + T_1), W_q^1(\Omega)), \end{aligned}$$

satisfying the estimates

$$\int_T^{T+T_1} \|\nabla \bar{\mathbf{u}}(\cdot, t)\|_{L_\infty(\Omega)} dt \leq \sigma/2, \quad \frac{1}{3}\rho_* < \rho_* + \bar{\theta}(x, t) < \frac{5}{3}\rho_* \quad ((x, t) \in \Omega \times (T, T + T_1)). \quad (4.20)$$

If we define θ_1, ω_1 , and \mathbf{u}_1 by

$$\begin{aligned} \theta_1(x, t) &= \begin{cases} \theta(x, t) & \text{for } 0 < t < T, \\ \bar{\theta}(x, t) & \text{for } T < t < T + T_1, \end{cases} & \mathbf{u}_1(x, t) &= \begin{cases} \mathbf{u}(x, t) & \text{for } 0 < t < T, \\ \bar{\mathbf{u}}(x, t) & \text{for } T < t < T + T_1, \end{cases} \\ \omega_1(x, t) &= \begin{cases} \omega(x, t) & \text{for } 0 < t < T, \\ \bar{\omega}(x, t) & \text{for } T < t < T + T_1, \end{cases} \end{aligned}$$

then θ_1 , ω_1 , and \mathbf{u}_1 solve (1.10) in $(0, T + T_1)$ and

$$\begin{aligned}\theta_1 &\in W_p^1((0, T + T_1), W_q^1(\Omega)), \mathbf{u}_1 \in L_p((0, T + T_1), W_q^2(\Omega)^N) \cap W_p^1((0, T + T_1), L_q(\Omega)^N), \\ \omega_1 &\in W_p^1((0, T + T_1), W_q^1(\Omega)).\end{aligned}$$

Moreover, by (4.16), (4.7), and (4.20), we have $\frac{1}{3}\rho_* < \rho_* + \theta_1(x, s) < \frac{5}{3}\rho_*$ and

$$\begin{aligned}\sup_{0 < t < T + T_1} \left\| \int_0^t \nabla \mathbf{u}_1(\cdot, s) ds \right\|_{L_\infty(\Omega)} &\leq \int_0^T \|\nabla \mathbf{u}(\cdot, s)\|_{L_\infty(\Omega)} ds + \int_T^{T + T_1} \|\nabla \bar{\mathbf{u}}(\cdot, s)\|_{L_\infty(\Omega)} ds \\ &\leq M_3 \epsilon + \sigma/2\end{aligned}$$

with some constant M_3 independent of ϵ . Choosing $\epsilon > 0$ so small that $M_3 \epsilon \leq \sigma/2$, we see that θ_1 and \mathbf{u}_1 satisfy (4.3), replacing T by $T + T_1$. Therefore, we can prolong θ , \mathbf{u} , and ω to $(0, T + T_1)$. It follows from (4.17) that T_1 is independent of ϵ , so that we can prolong θ , \mathbf{u} , and ω to time interval $(0, \infty)$ finally with $\mathbf{I}(\infty) \leq r_1(\epsilon)$, which completes the proof of the existence part of Theorem 1.1. But, the uniqueness follows from the local in time unique existence theorem (Theorem 1.1), which completes the proof of Theorem 1.1.

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