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Fractional Derivative and Financial Instruments: Waiting Time Distributions for the Exchange Rate Movement of US Dollar to Japanese Yen

Bambang Hendriya Guswanto, Muhammad Rizki Pratama, Agung Prabowo, Jajang Jajang and Idha Sihwaningrum*

Department of Mathematics, Jenderal Soedirman University, Purwokerto, Indonesia

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Abstract: In this article, a mathematical model for exchange rate movements is derived by applying random walks theory and involving Caputo fractional derivative operator. The waiting time distributions for the exchange rate movement of US Dollar to Japanese Yen, which are intimately related to the mathematical model, during February 2019 are also studied. Three types of waiting time distributions, i.e., exponential, stretched exponential, and Mittag-Leffler distributions are compared. The result shows that Mittag-Leffler Distribution is the best distribution to approximate the empirical distribution of the exchange rate data during February 2019 except the data of February 18, 2019 which is approximated better by stretched exponential distribution.

Keywords: Exchange rate, US Dollar, Japanese Yen, stretched exponential distribution, Mittag-Leffler distribution.

1 Introduction

Many researches employed nonpoissonian waiting time distributions to analyze financial instruments. In [1], Mainardi et al studied the distribution of Germany government obligation transaction data approximated by Mittag-Leffler distribution. Raberto et al in [2] used stretched exponential distribution to model stock price movement data. Sabateli et al in [3] approximated the waiting time of 10 stock prices in Ireland stock exchange by using Mittag-Leffler distribution. In [4], random walks theory was used to analyze financial instrument value waiting time by using two types of survival distribution, i.e., exponential and Mittag-Leffler distribution. Scalas et al in [5] modelled the waiting time of 30 stock prices of Dow Jones Industrial Average in American stock exchange by using non-exponential distribution.

Here, a mathematical model for exchange rate movements is derived by applying random walks theory and involving Caputo fractional derivative operator. The waiting time distributions for the exchange rate movement of US Dollar to Japanese Yen during February 2019 are also studied. The waiting time distributions are intimately related to the mathematical model. Three types of waiting time distributions, i.e., exponential, stretched exponential, and Mittag-Leffler distributions are compared. The result shows that Mittag-Leffler Distribution is the best distribution in approximating the empirical distribution for the exchange rate data during February 2019 except the empirical distribution for the exchange rate data on February 18, 2019 which is approximated better by stretched exponential distribution.

This article is composed of four sections. The second section contains the derivation of fractional Kolmogorov-Feller equation that can be used to analyze exchange rate movements in value x and time t derived from random walks process. In the third section, we give the analysis of waiting time distributions for the exchange rate movement of US Dollar to Japanese Yen during February 2019. In the last section, a conclusion and future work are given.

* Corresponding author e-mail: bambang.guswanto@unsoed.ac.id

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2 Mathematical modelling

In this section, we derive mathematical models that can be used to analyze exchange rate movements in value x and time t. We derive it from random walks process. We suppose that $\lambda(x,y) = \lambda(y-x)$ and $\psi(t)$ stand for the probability density that the exchange rate moves from a value x to a value y and the probability density that the exchange rate moves after a waiting time t, respectively. We assume that the probability density $\lambda(x,y)$ is independent of the probability density $\psi(t)$. Following the way by Othmer et al. in [6] applied to this financial instrument, we denote the conditional probability that the exchange rate reaches a value x at time t after k jumps by $J_k(x,t)$, that is

$$J_k(x,t) = \int_0^t \int_{-\infty}^{\infty} \psi(t-\tau)\lambda(x-y)J_{k-1}(y,\tau)dyd\tau. \tag{1}$$

If $J(x,t)$ is the probability density that the exchange rate reaches x at t then

$$\begin{aligned} J(x,t) &= \sum_{k=0}^{\infty} J_k(x,t) \\ &= \delta(x)\delta(t) + \int_0^t \int_{-\infty}^{\infty} \psi(t-\tau)\lambda(x-y)J(y,\tau)dyd\tau \end{aligned} \tag{2}$$

where $J_0(x,t) = \delta(x)\delta(t)$.

We next denote the probability that the exchange rate value is x at t by $p(x,t)$ with the initial condition $x_0 = 0$. Then

$$p(x,t) = \int_0^t \Psi(t,\tau : x)J(x,\tau)d\tau \tag{3}$$

where $\Psi(t,\tau : x) = \Psi(t-\tau)$ denotes the probability density that the exchange rate value is x at $t < \tau$ and does not move during the time interval $t-\tau$. Therefore

$$\Psi(t) = \int_t^{\infty} \psi(\tau)d\tau = 1 - \int_0^t \psi(\tau)d\tau. \tag{4}$$

By substituting (2) into (3), we have

$$p(x,t) = \delta(x)\Psi(t) + \int_0^t \int_{-\infty}^{\infty} \psi(t-\tau)\lambda(x-y)p(y,\tau)dyd\tau \tag{5}$$

where $p(x,0) = \delta(x)$. By using Laplace and Fourier transforms

$$\tilde{f}(s) = \int_0^t e^{-st} f(t)dt, \quad \hat{f}(k) = \int_{-\infty}^{\infty} e^{-2\pi ikx} f(x)dx \tag{6}$$

and employing some algebraic operations applied to (5), we get

$$\tilde{\Phi}(s)[s\tilde{p}(x,s) - 1] = \hat{\lambda}(k)\hat{\tilde{p}}(k.s) - \hat{\tilde{p}}(k.s) \tag{7}$$

where $\tilde{\Phi}(s) = \tilde{\Psi}(s)/\tilde{\psi}(s)$. By the inverse of Laplace and Fourier Transforms applied to (7), it follows that

$$\int_0^t \Phi(t-\tau)\frac{\partial}{\partial\tau}p(x,\tau)d\tau = \int_{-\infty}^{\infty} \lambda(x-y)p(y,t)dy - p(x,t). \tag{8}$$

If the probability density of waiting time is exponential function, i.e.,

$$\psi(t) = \frac{1}{\gamma}e^{-\frac{t}{\gamma}}, \ \gamma > 0 \tag{9}$$

then $\tilde{\Phi}(s) = \gamma$. By substituting this $\tilde{\Phi}(s)$ into (7) and then inverting it back to space-time domain, we obtain

$$\frac{\partial}{\partial t}p(x,t) = \frac{1}{\gamma}\int_{-\infty}^{\infty} \lambda(x-y)[p(y,t) - p(x,t)]dy. \tag{10}$$

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The equation (10) is well known Kolmogorov-Feller equation. The survival and cumulative hazard functions of (9) are

$$\Psi(t) = e^{-\frac{t}{\gamma}} \tag{11}$$

and

$$\Lambda = \frac{t}{\gamma}, \tag{12}$$

respectively.

If the probability density of waiting time is

$$\psi_\alpha(t) = \frac{t^{\alpha-1}}{\gamma^\alpha} E_{\alpha,1}\left(-\left(\frac{t}{\gamma}\right)^\alpha\right), \; 0<\alpha<1, \; \gamma>0, \tag{13}$$

where $E_{\alpha,\beta}(t)$ is Mittag-Leffler function defined by

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{14}$$

then $\tilde{\Phi}(s) = \gamma^\alpha s^{\alpha-1}$. Again, by substituting this $\tilde{\Phi}(s)$ into (7) and then inverting it back to space-time domain, we obtain

$$\frac{\partial^\alpha}{\partial t^\alpha} p(x,t) = \frac{1}{\gamma^\alpha} \int_{-\infty}^{\infty} \lambda(x-y)[p(y,t)-p(x,t)]dy \tag{15}$$

where d^α/dt^α is Caputo fractional derivative defined by

$$\frac{d^\alpha}{dt^\alpha} f(t) = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{d\tau} f(\tau) d\tau. \tag{16}$$

The equation (15) is called fractional Kolmogorov-Feller equation. The survival and cumulative hazard functions of (13) are

$$\Psi_\alpha(t) = E_{\alpha,1}\left(-\left(\frac{t}{\gamma}\right)^\alpha\right), \; 0<\alpha<1, \; \gamma>0 \tag{17}$$

and

$$\Lambda_\alpha(t) = -\ln E_{\alpha,1}\left(-\left(\frac{t}{\gamma}\right)^\alpha\right), \tag{18}$$

respectively. Note that if $\alpha = 1$, (13) is reduced to (9). Thus, Mittag-Leffler distribution and fractional Kolmogorov-Feller equation become exponential distribution and Kolmogorov-Feller equation, respectively. If we then take only the first two terms of (17) then

$$E_{\alpha,1}\left(-\left(\frac{t}{\gamma}\right)^\alpha\right) \approx 1 - \frac{t^\alpha}{\gamma^\alpha \Gamma(1+\alpha)} \approx e^{-\frac{t^\alpha}{\gamma^\alpha \Gamma(1+\alpha)}} \tag{19}$$

which is called stretched exponential function and we write

$$\Psi_\alpha^S = e^{-\frac{t^\alpha}{\gamma^\alpha \Gamma(1+\alpha)}} \tag{20}$$

The cumulative hazard function associated with (20) is

$$\Lambda_\alpha^S = \frac{t^\alpha}{\gamma^\alpha \Gamma(1+\alpha)}. \tag{21}$$

In next section, we compare (11), (17), and (20) with the empirical survival distribution for the exchange rate data of US Dollar to Japanese Yen during February 2019. We also compare (12), (18), and (21) with the empirical cumulative hazard distribution for the exchange rate data.

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3 Waiting time distributions

In this section, we give some waiting time distributions for the exchange rate movement of US Dollar to Japanese Yen during February 2019. We use the exchange rate movement data of USD to JPY per time unit of second during trade hours opened from February 1 to 28, 2019. The data are obtained from [7]. We first compute the waiting times of the exchange rates movement by using Microsoft Excel programming and then process them by Matlab.

Following Bain and Engelhardt [8], we determine the waiting time distribution for the empirical data as follows. If observation data $t_1, t_2, \ldots, t_n$ are given and y_i, $i = 1, 2, \ldots, n$ are the data after sorting from the least value to the largest value then the empirical cumulative distribution $\psi(t)$ of the data is given by

$$\psi(t) = \begin{cases} 0, & t < y_1, \\ \frac{i}{n}, & y_i < t < y_{i+1} \\ 1, & t \geq y_n \end{cases} \tag{22}$$

where n, t, and y_i stand for number of observations, the waiting time of the exchange rate movement, and i-th observation after the data $t_1, t_2, \ldots, t_n$ are sorted from the least value to the largest value, respectively. Therefore, we have the survival distribution $\Psi(t)$ of $\psi(t)$, that is

$$\Psi(t) = \begin{cases} 1, & t < y_1, \\ 1 - \frac{i}{n}, & y_i < t < y_{i+1}, \\ 0, & t \geq y_n. \end{cases} \tag{23}$$

To determine Mittag-Leffler distribution estimations, we use the Matlab programmings by Garappa [9] and Podlubny [10]. Table 1 below represents the parameters values of some waiting time distributions for the exchange rate movement during February 2019.

Table 1: Distributions parameters

Date	(Stretched) Exponential Distribution	Mittag-Leffler Distribution	
	γ	α	γ
February 1, 2019	3.9798	0.8825	2.4026
February 3, 2019	4.8126	0.8792	2.9608
February 4, 2019	5.2513	0.8782	3.2948
February 5, 2019	4.6169	0.8811	2.8817
February 6, 2019	4.6946	0.8906	3.0377
February 7, 2019	3.6694	0.8926	2.2358
February 8, 2019	4.4332	0.8978	2.9386
February 10, 2019	6.4762	0.8785	4.3212
February 11, 2019	4.3438	0.8864	2.6948
February 12, 2019	5.0975	0.8934	3.3961
February 13, 2019	4.849	0.8926	3.1626
February 14, 2019	3.7803	0.8792	2.2234
February 15, 2019	4.0236	0.8949	2.5624
February 17, 2019	5.7664	0.8651	3.4318
February 18, 2019	5.9816	0.8813	3.9797
February 19, 2019	4.671	0.8981	3.1223
February 20, 2019	4.7577	0.8899	3.0528
February 21, 2019	4.4653	0.8926	2.824
February 22, 2019	4.8803	0.8921	3.2419
February 24, 2019	4.6235	0.8606	2.6691
February 25, 2019	4.6788	0.8944	3.0603
February 26, 2019	5.0985	0.887	3.3017
February 27, 2019	4.6941	0.8822	2.9239
February 28, 2019	4.3893	0.875	2.6435

The following figures represent the comparison between Mittag-Leffler, exponential, stretched exponential distributions and empirical survival distribution on February 1, 3, and 4, 2019.

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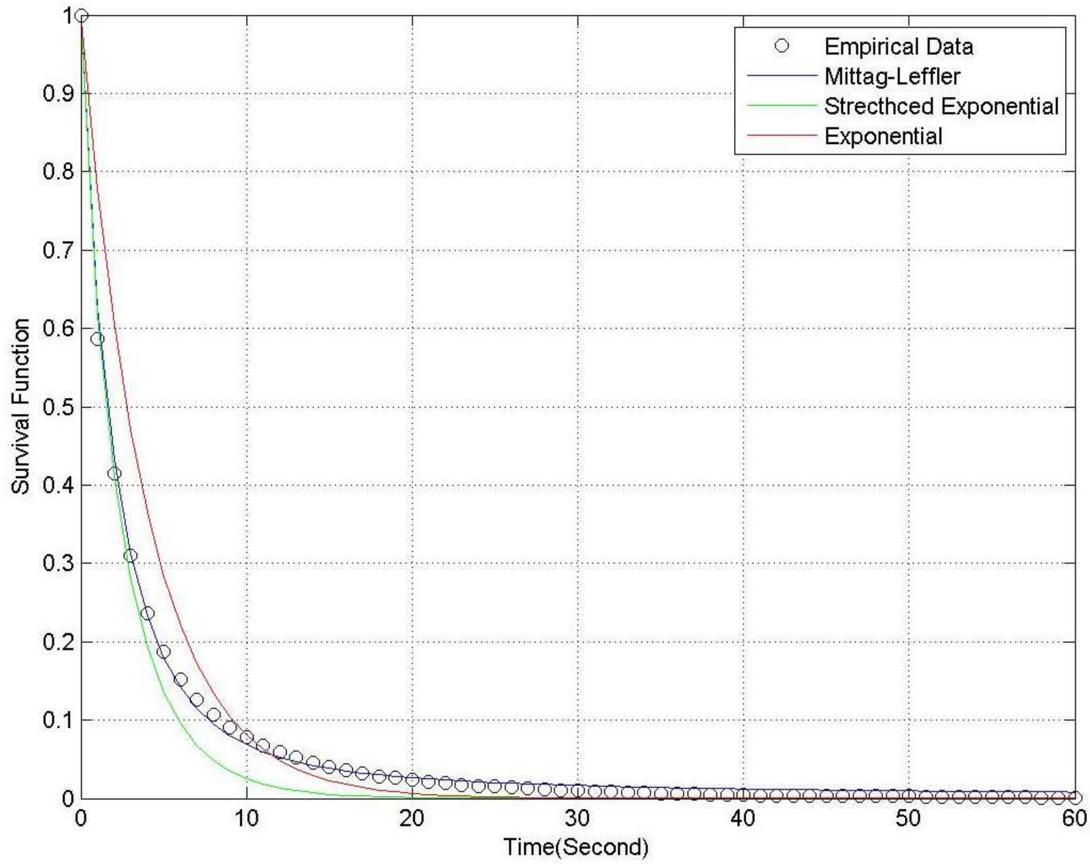

Fig. 1: Exponential, Mittag-Leffler, stretched exponential survival, and empirical survival distributions on February 1, 2019

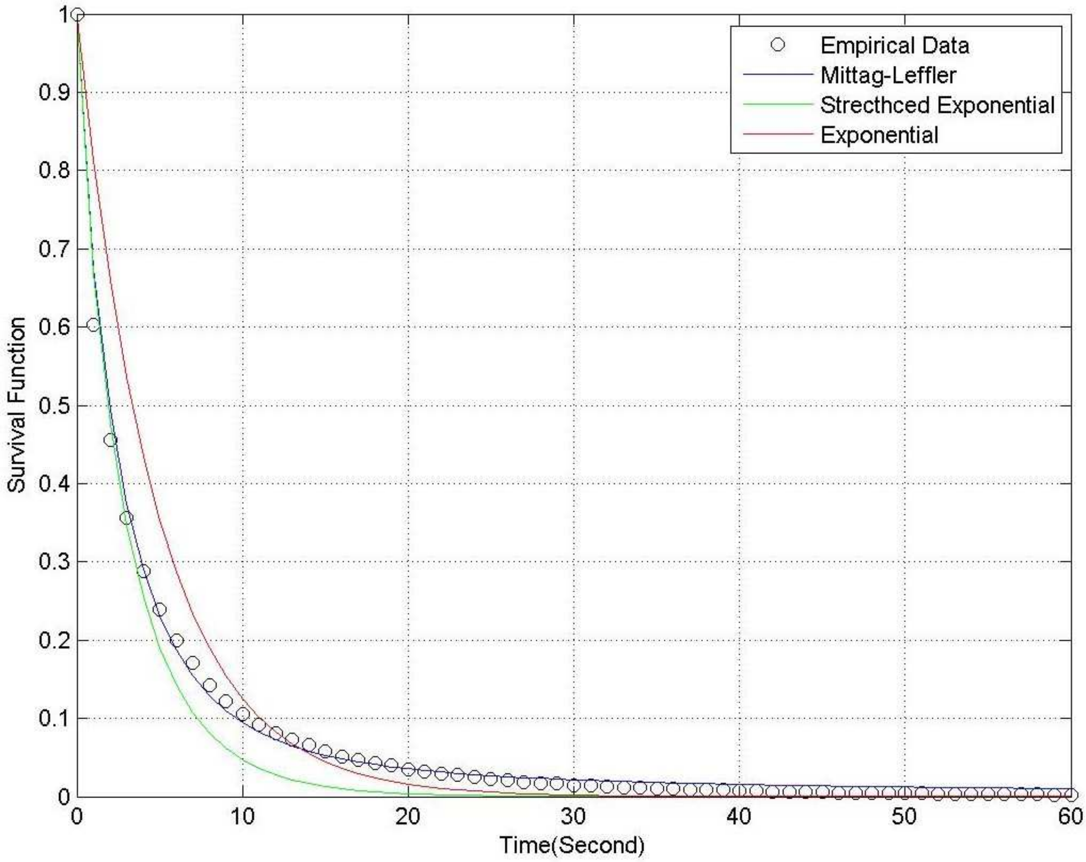

Fig. 2: Exponential, Mittag-Leffler, stretched exponential, and empirical survival distributions on February 3, 2019

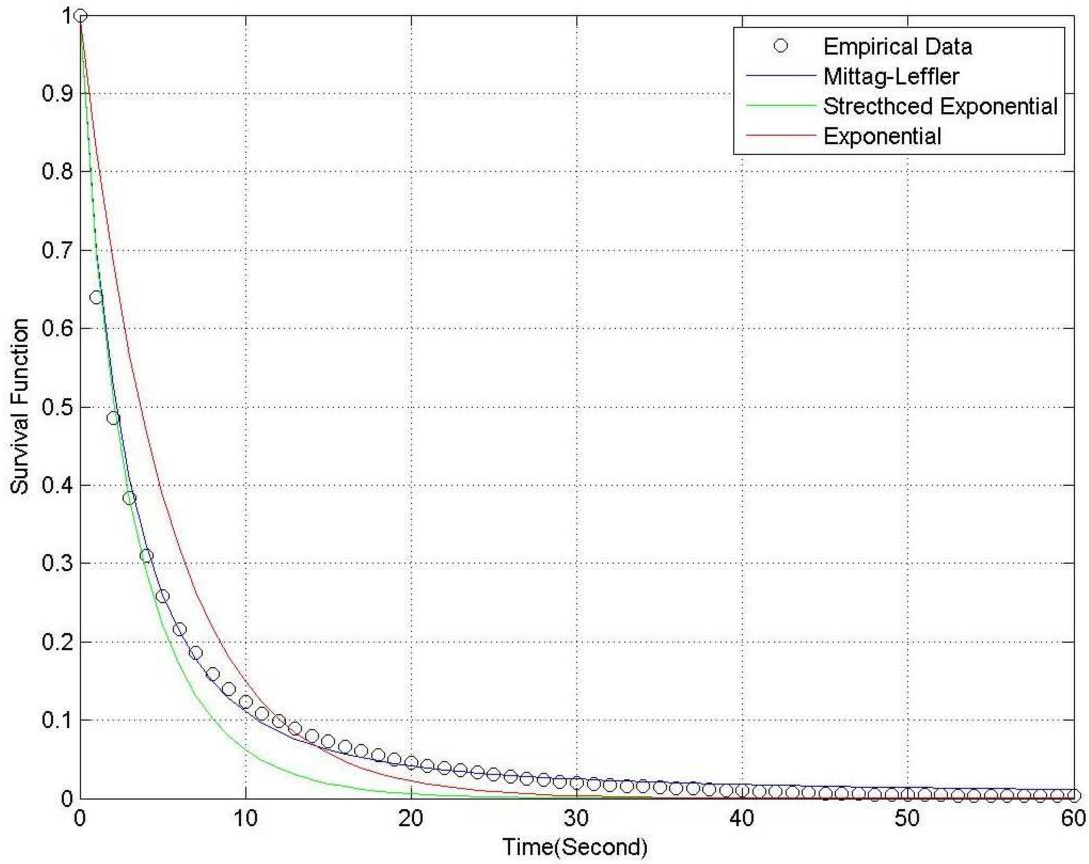

Fig. 3: Exponential, Mittag-Leffler, stretched exponential, and empirical survival distributions on February 4, 2019

From Figure 1, 2, and 3, we observe that Mittag-Leffer survival distribution approximates the empirical survival distribution better than the other survival distributions. This fact also occurs for the exchange rate movement on the other dates in February 2019. We then determine the best survival distribution by comparing the mean abolute errors between the empirical survival distribution and Mittag-Leffler, exponential, and stretched survival distributions. The following table provides them.

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Table 2: Mean Absolute Errors

Date	Mean Absolute Errors		
	Mittag-Leffler	Stretched Exponential	Exponential
February 1, 2019	0.0064	0.0099	0.0124
February 3, 2019	0.0051	0.0055	0.0066
February 4, 2019	0.0062	0.0069	0.0082
February 5, 2019	0.0069	0.0091	0.0109
February 6, 2019	0.0065	0.0105	0.0129
February 7, 2019	0.0051	0.0064	0.0084
February 8, 2019	0.0067	0.0095	0.0118
February 10, 2019	0.0082	0.0112	0.0129
February 11, 2019	0.0050	0.0056	0.0068
February 12, 2019	0.0062	0.0072	0.0088
February 13, 2019	0.0062	0.0092	0.0113
February 14, 2019	0.0061	0.0070	0.00881
February 15, 2019	0.0062	0.0091	0.0117
February 17, 2019	0.0069	0.0124	0.0149
February 18, 2019	0.0053	0.0042	0.0049
February 19, 2019	0.0060	0.0076	0.0094
February 20, 2019	0.0054	0.0063	0.0077
February 21, 2019	0.0048	0.0059	0.0074
February 22, 2019	0.0068	0.0086	0.0104
February 24, 2019	0.0083	0.0141	0.0165
February 25, 2019	0.0052	0.0057	0.0070
February 26, 2019	0.0060	0.0070	0.0083
February 27, 2019	0.0056	0.0061	0.0074
February 28, 2019	0.0067	0.0082	0.0098

From Table 2, we observe that mean absolute error between Mittag-Leffler and empirical survival distributions has the least value on all dates except February 18, 2019. On the date, mean value error between stretched exponential and empirical survival distributions has the least value. Thus, Mittag-Leffler survival distribution is the best survival distribution among the existing distributions.

We next compare the cumulative hazard rates of the ditributions as shown by figures 4-6.

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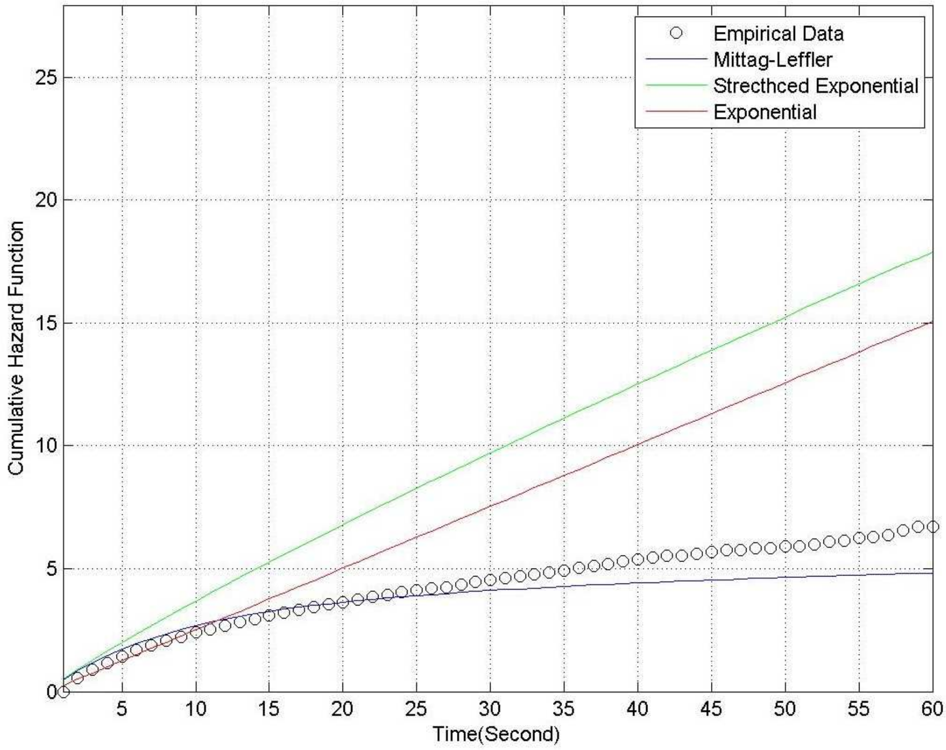

Fig. 4: The cumulative hazard rates of exponential, Mittag-Leffler, stretched exponential, and empirical distributions on February 1, 2019

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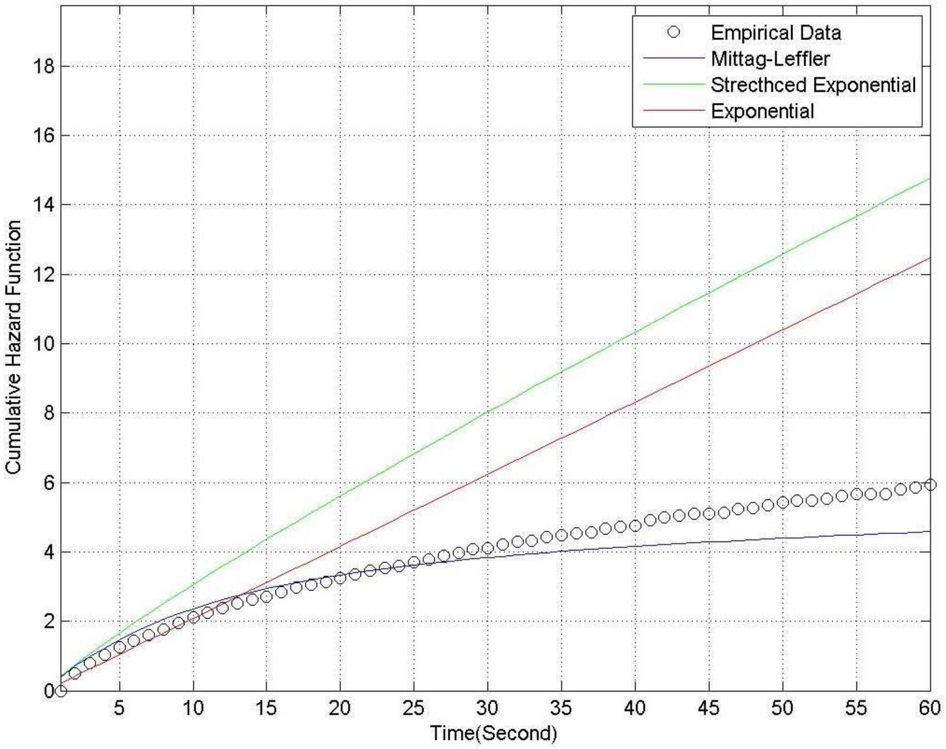

Fig. 5: The cumulative hazard rates of exponential, Mittag-Leffler, stretched exponential, and empirical distributions on February 3, 2019

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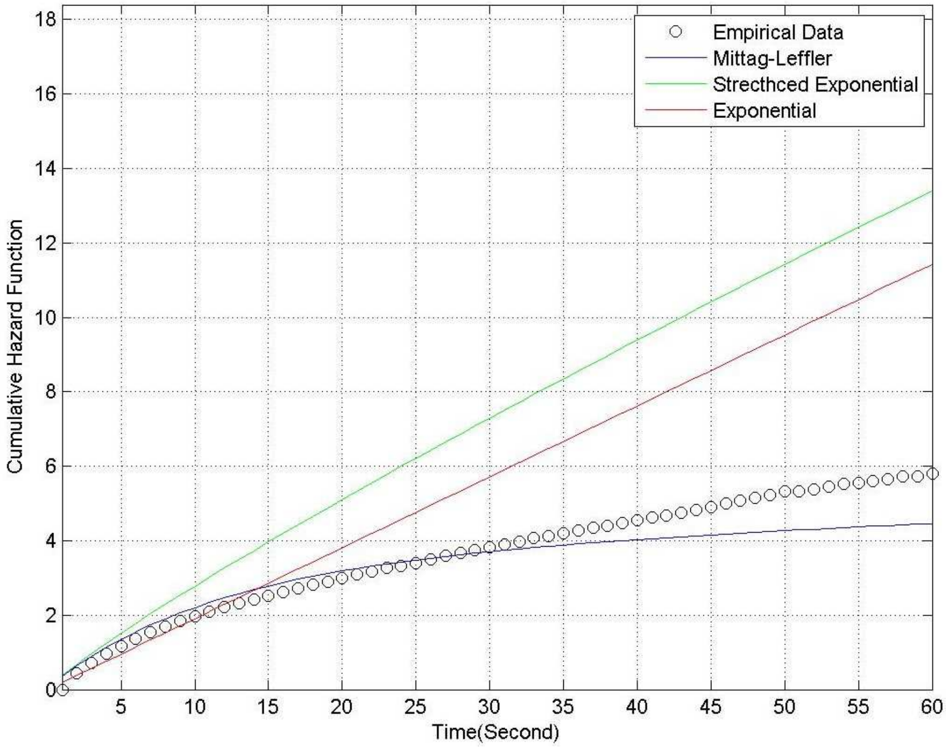

Fig. 6: The cumulative hazard rates of exponential, Mittag-Leffler, stretched exponential, and empirical distributions on February 4, 2019

Based on Figure 4, 5, and 6, we observe that the cumulative hazard rate of the empirical distribution is approximated better by the cumulative hazard rate of Mittag-Leffler distribution than the cumulative hazard rates of the other distributions. However, the cumulative hazard rate of Mittag-Leffler distribution is sufficiently close to the cumulative hazard rate of the empirical distribution on the time interval $(0,30]$. These facts also occur for the exchange rate movement on the other dates in February 2019.

4 Conclusion

We find that Mittag-Leffler function gives better waiting time distribution for the exchange rate movement of US Dollar to Japanese Yen during February 2019 if it is compared with exponential and stretched exponential functions. There is a satisfactory agreement between Mittag-Leffler distribution and the empirical distribution, especially on the time interval $(0,30]$. After 30 seconds, Mittag-Leffler survival function tends to 0. It means that the probability that the exchange rate does not move after the waiting time more than 30 seconds is sufficiently low. In other words, the probability that the exchange rate moves after the waiting time 30 seconds is sufficiently high. Practitioners in currency trades should not take a decision to sell or buy US Dollar to Japanes Yen after the waiting time less than 30 seconds since the probability that the exchange rate moves after the waiting time less than 30 seconds is sufficiently low.

Mittag Leffler waiting time distribution as explained in previous section is associated with fractional Kolmogorov-Feller equation derived from random walks process. Thus, for future work on this topic, the analysis of the probability for the exchange rate movement in value x is required so that fractional Kolmogorov-Feller equation can be used to analyze the exchange rate movement not only in time t but also in value x.

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