free boundary

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On the free boundary problem for the Oldroyd-B Model in the maximal L_p–L_q regularity class

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ABSTRACT

In the present work, we prove the local well-posedness of non-Newtonian compressible viscous barotropic fluid flow of Oldroyd-B type with free surface in a bounded domain of N-dimensional Euclidean space ($N \geq 2$). The key step is to prove the maximal L_p–L_q regularity theorem for the linearized equation with the help of the $\mathcal{R}$-bounded solution operators for the corresponding resolvent problem and Weis's operator valued Fourier multiplier theorem.

1. Introduction and main result

Let Ω be a bounded domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) whose boundary consists of two parts Γ_0 and Γ_1, where $\Gamma_0 \cap \Gamma_1 = \emptyset$. The Ω is occupied by a compressible viscous barotropic non-Newtonian fluid of Oldroyd-B type. The present paper deals with the problem of determining the region $\Omega_t \subset \mathbb{R}^N$, the density field $\rho = \rho(x,t)$, the elastic tensor $\tau = \tau(x,t)$, and the velocity field $\mathbf{u} = (u_1(x,t), \ldots, u_N(x,t))$, which satisfy the system of equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 & \text{in } \Omega_t, \\ \rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \operatorname{Div}\mathbf{T}(\mathbf{u}, P(\rho)) = \beta \operatorname{Div}\tau & \text{in } \Omega_t, \\ \partial_t \tau + \mathbf{u}\cdot\nabla\tau + \gamma\tau = \delta\mathbf{D}(\mathbf{u}) + g_\alpha(\nabla\mathbf{u}, \tau) & \text{in } \Omega_t, \\ (\mathbf{T}(\mathbf{u}, P(\rho)) + \beta\tau)\mathbf{n}_t = -P(\rho_*)\mathbf{n}_t & \text{on } \Gamma_t, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \\ (\rho, \mathbf{u}, \tau)|_{t=0} = (\rho_* + \theta_0, \mathbf{u}_0, \tau_0) & \text{in } \Omega, \\ \Omega_t|_{t=0} = \Omega_0, \quad \Gamma_t|_{t=0} = \Gamma_1 \end{cases} \tag{1.1}$$

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for $0 < t < T$. Here, ρ_* is a positive constant describing the mass density of the reference domain Ω, $\mathbf{T}(\mathbf{u}, P(\rho))$ the stress tensor of the form

$$\mathbf{T}(\mathbf{u}, \rho) = \mathbf{S}(\mathbf{u}) - P(\rho)\mathbf{I} \quad \text{with } \mathbf{S}(\mathbf{u}) = \mu\mathbf{D}(\mathbf{u}) + (\nu - \mu)\text{div}\, \mathbf{u}\mathbf{I}, \tag{1.2}$$

$\mathbf{D}(\mathbf{u})$ the doubled deformation tensor whose (i, j) components are $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$ $(\partial_i = \partial/\partial x_j)$, $\mathbf{I}$ the $N \times N$ identity matrix, μ, ν, β, γ and δ are positive constants (μ and ν are the first and second viscosity coefficients, respectively), $\mathbf{n}_t$ is the unit outer normal to Γ_t, $P(\rho)$ a C^∞ function defined for $\rho > 0$ which satisfies that $P'(\rho) > 0$ for $\rho > 0$. Moreover, the function $g_\alpha(\nabla\mathbf{u}, \tau)$ has a form

$$g_\alpha(\nabla\mathbf{u}, \tau) = \mathbf{W}(\mathbf{u})\tau - \tau\mathbf{W}(\mathbf{u}) + \alpha(\tau\mathbf{D}(\mathbf{u}) + \mathbf{D}(\mathbf{u})\tau), \tag{1.3}$$

where α is a constant with $-1 \leq \alpha \leq 1$ and $\mathbf{W}(\mathbf{u})$ the doubled antisymmetric part of the gradient $\nabla\mathbf{u}$ whose (i, j) components are $W_{ij}(\mathbf{u}) = \partial_i u_j - \partial_j u_i$. Finally, for any matrix field $\mathbf{K}$ whose components are K_{ij}, the quantity $\text{Div}\, \mathbf{K}$ is an N vector whose ith component is $\sum_{j=1}^N \partial_j K_{ij}$, and also for any vector of functions $\mathbf{u} = (u_1, \ldots, u_N)$, $\text{div}\, \mathbf{u} = \sum_{j=1}^N \partial_j u_j$, and $\mathbf{u} \cdot \nabla\mathbf{u}$ is an N vector whose ith component is $\sum_{j=1}^N u_j \partial_j u_i$. We assume that the boundary of Ω_t consists of Γ_0 and Γ_t with $\Gamma_0 \cap \Gamma_t = \emptyset$.

Aside from the dynamical system (1.1), a further kinematic condition for Γ_t is satisfied, which gives

$$\Gamma_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in \Gamma_1)\}, \tag{1.4}$$

where $\mathbf{x} = \mathbf{x}(\xi, t)$ is the solution to the Cauchy problem:

$$\Gamma_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in \Gamma_1)\}. \tag{1.5}$$

Concerning the free boundary problem of the viscous compressible barotropic Newtonian fluid flow, the local well-posedness and global well-posedness have been studied in the L_2 Sobolev–Slobodetskii space by Denisova and Solonnikov [4,3], Secchi and Valli [17–19], Solonnikov and Tani [28,30,31], and Zajaczkowski [34,35], and in the L_p–L_q maximal regularity class by Shibata et al. [7,24]. Recently, M. Nesensohn [14] proved the local well-posedness of the free boundary problem for the non-Newtonian fluid flow of Oldroyd-B type in the incompressible viscous fluid case (further references are found in [14]). On the other hand, Shi, Wang and Zhang [20] investigated the asymptotic stability for 1-dimensional motion of non-Newtonian compressible fluids using L_2 energy method. Meanwhile, global existence of strong solutions of Navier–Stokes equations with non-Newtonian potential for 1-dimensional isentropic compressible fluids has been studied by Liu, Yuan and Lie [9]. The purpose of this paper is to study the local well-posedness of problem (1.1).

To prove the local well-posedness of problem (1.1), we use the Lagrangian coordinate in order to transform the time dependent domain Ω_t to the fixed domain Ω. Let $\mathbf{u}(x, t)$ and $\mathbf{v}(\xi, t)$ be velocity fields in the Euler coordinate and in the Lagrangian coordinate, respectively. The Euler coordinate system $\{x\}$ and Lagrangian coordinate system $\{\xi\}$ are connected by the relation:

$$x = \xi + \int_0^t \mathbf{v}(\xi, s)ds \equiv \mathbf{X}_\mathbf{v}(\xi, t),$$

where, $\mathbf{v}(\xi, t) = (v_1(\xi, t), \ldots, v_N(\xi, t)) = \mathbf{u}(\mathbf{X}_\mathbf{v}(\xi, t), t)$. Let A be the Jacobi matrix of the transformation $x = \mathbf{X}_\mathbf{v}(\xi, t)$, whose (i, j) element is $a_{ij} = \delta_{ij} + \int_0^t (\frac{\partial v_i}{\partial \xi_j})(\xi, s)ds$. There exists a small number σ such that if

$$\max_{i,j=1,\ldots,N} \left\| \int_0^t \frac{\partial v_i}{\partial \xi_j}(\cdot, s)ds \right\|_{L_\infty(\Omega)} < \sigma \quad (0 < t < T), \tag{1.6}$$

then A is invertible, that is, $\det A \neq 0$. Thus, we have $\nabla_x = A^{-1}\nabla_\xi = (\mathbf{I} + \mathbf{V}_0(\int_0^t \nabla\mathbf{v}(\xi, s)ds))\nabla_\xi$, where $\mathbf{V}_0(\mathbf{K})$ is an $N \times N$ matrix of C^∞ functions with respect to $\mathbf{K} = (k_{ij})$ for $|\mathbf{K}| < 2\sigma$ and $\mathbf{V}_0(0) = 0$. Here and hereafter, k_{ij} denote corresponding variables to $\int_0^t (\frac{\partial v_i}{\partial \xi_j})(\cdot, s)ds$. Let $\mathbf{n}$ be the unit outward normal to

Γ_0, and then we have

$$\mathbf{n}_t = \frac{A^{-1}\mathbf{n}}{|A^{-1}\mathbf{n}|}. \tag{1.7}$$

Suppose that $\rho(x,t)$, $\tau(x,t)$ and $\mathbf{u}(x,t)$ are solutions of (1.1). Setting $\rho(\mathbf{X_v}(\xi,t),t) = \rho_* + \theta_0(\xi) + \theta(\xi,t)$ and $\tau = \tau_0(\xi) + \pi(\xi,t)$, we see that problem (1.1) is transformed to the following equations:

$$\begin{cases} \theta_t + (\rho_* + \theta_0)\operatorname{div}\mathbf{v} = F(\theta,\mathbf{v},\pi) & \text{in } \Omega\times(0,T),\\ (\rho_*+\theta_0)\mathbf{v}_t - \operatorname{Div} S(\mathbf{v}) + \nabla(P'(\rho_*+\theta_0)\theta) = \mathbf{g} + \beta\operatorname{Div}\pi + \mathbf{G}(\theta,\mathbf{v},\pi) & \text{in } \Omega\times(0,T),\\ \pi_t + \gamma\pi - g_\alpha(\nabla\nu,\tau_0) - \delta\mathbf{D}(\mathbf{v}) = -\gamma\tau_0 + \mathbf{L}(\theta,\mathbf{v},\pi) & \text{in } \Omega\times(0,T),\\ (\mathbf{S}(\mathbf{v}) - P'(\rho_*+\theta_0)\theta\mathbf{I} + \beta\pi)\mathbf{n} = \mathbf{h} + \mathbf{H}(\theta,\mathbf{v},\pi) & \text{on } \Gamma_1\times(0,T),\\ \mathbf{v} = 0 & \text{on } \Gamma_0\times(0,T),\\ (\theta,\mathbf{v},\pi)|_{t=0} = (0,\mathbf{u}_0,0) & \text{in } \Omega, \end{cases} \tag{1.8}$$

where $\mathbf{g} = -P'(\rho_*+\theta_0)\nabla\theta_0 + \beta\operatorname{Div}\tau_0$ and $\mathbf{h} = \big(P(\rho_*+\theta_0) - P(\rho_*)\big)\mathbf{n} - \beta\tau_0\mathbf{n}$. Moreover, $F(\theta,\mathbf{v})$, $\mathbf{G}(\mathbf{v},\theta,\pi)$, $\mathbf{L}(\mathbf{v},\pi)$, and $\mathbf{H}(\mathbf{v},\theta,\pi)$ are nonlinear functions of the forms:

$$F(\theta,\mathbf{v}) = -\theta\operatorname{div}\mathbf{v} - (\rho_*+\theta_0+\theta)V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{v}, ds\right)\nabla\mathbf{v},$$

$$\begin{aligned}
\mathbf{G}(\mathbf{v},\theta,\pi) &= -\theta\mathbf{v}_t + \operatorname{Div}\left(\mu V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} + (\nu-\mu)V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\mathbf{I}\right)\\
&+ V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\left(\mu\left(D(\mathbf{v}) + V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right) + (\nu-\mu)\left(\operatorname{div}\mathbf{v} + V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right)\mathbf{I}\right)\\
&- P'(\rho_*+\theta_0+\theta)V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla(\theta_0+\theta) + \beta V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\tau_0 + \beta V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\pi\\
&- \nabla\left(\int_0^1 P''(\rho_*+\theta_0+\ell\theta)(1-\ell)\,d\ell\theta^2\right),
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}(\mathbf{v},\theta,\pi) &= -\left\{\mu V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} + (\nu-\mu)\left(V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right)\mathbf{I}\right\}\mathbf{n}\\
&- \left\{\mu\left(D(\mathbf{v}) + V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right) + (\nu-\mu)\left(\operatorname{div}\mathbf{v} + V_{\mathrm{div}}\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\right)\mathbf{I}\right\}\\
&\times V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\mathbf{n} + \left(\int_0^1 P''(\rho_*+\theta_0+\ell\theta)(1-\ell)d\ell\theta^2\right)\mathbf{n} + (P(\rho_*+\theta_0+\theta) - P(\rho_*))\\
&\times V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\mathbf{n} - \beta(\tau_0+\pi)V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\mathbf{n}
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}(\mathbf{v},\pi) &= \mathbf{W}(\mathbf{v})\tau + V_W\left(\int_0^t\nabla\mathbf{v}ds\right)\nabla\mathbf{v}(\tau_0+\tau) - \tau\mathbf{W}(\mathbf{v}) - (\tau+\tau_0)V_W\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}\\
&+ \alpha(\tau\mathbf{D}(\mathbf{v}) + (\tau+\tau_0)V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v} + \mathbf{D}(\mathbf{v})\tau + V_D\left(\int_0^t\nabla\mathbf{v}\,ds\right)\nabla\mathbf{v}(\tau+\tau_0)),
\end{aligned}$$

and $V_D(\mathbf{K})$, $V_W(\mathbf{K})$, and $V_{\mathrm{div}}(\mathbf{K})$ are some matrices of C^∞ functions with respect to $\mathbf{K}$ for $|\mathbf{K}| \le 2\sigma$, which satisfy the condition:

$$V_D(0) = 0, \qquad V_W(0) = 0, \qquad V_{\mathrm{div}}(0) = 0. \tag{1.9}$$

Employing the argumentation due to Ströhmer [32], we can show eventually that the correspondence $x = X_v(\xi,t)$ is invertible, then problem (1.1) and problem (1.8) are equivalent. Thus, we show the local well-posedness of problem (1.8).

To this end, the main step is to prove the L_p–L_q maximal regularity for the following linearized problem:

$$\begin{cases} \partial_t\rho + \gamma_1 \mathrm{div}\,\mathbf{u} = f & \text{in } \Omega\times(0,T), \\ \gamma_2\partial_t\mathbf{u} - \mathrm{Div}\,\mathbf{T}(\mathbf{u},\gamma_3\rho) = \delta_1 \mathrm{Div}\,\tau + \mathbf{g} & \text{in } \Omega\times(0,T), \\ \partial_t\tau + \delta_2\tau - g_\alpha(\nabla\mathbf{u},\tau_1) = \delta_3\mathbf{D}(\mathbf{u}) + \mathbf{h} & \text{in } \Omega\times(0,T), \\ (\mathbf{T}(\mathbf{u},\gamma_3\rho) + \delta_1\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \\ (\rho,\mathbf{u},\tau)|_{t=0} = (\rho_0,\mathbf{u}_0,\tau_0) & \text{in } \Omega, \end{cases} \tag{1.10}$$

where γ_1, γ_2, γ_3 and τ_1 are uniformly continuous functions with respect to $x \in \overline{\Omega}$, which satisfy the assumptions:

$$\begin{aligned} &\rho_*/2 \le \gamma_2(x) \le 2\rho_*, \quad 0 \le \gamma_1(x), \qquad \gamma_3(x) \le \rho_1, \qquad \|\nabla\gamma_\ell\|_{L_r(\Omega)} \le \rho_1, \quad (\ell = 1,2,3), \\ &\|\tau_1\|_{W^1_r(\Omega)} \le \rho_1, \end{aligned} \tag{1.11}$$

while δ_1, δ_2, and δ_3 are positive constants. Note that in problem (1.1) we have written $\delta_1 = \beta$, $\delta_2 = \gamma$ and $\delta_3 = \delta$.

The maximal L_p regularity was proved by Solonnikov [26,27] for the general parabolic equations which satisfy the uniform Lopatinski–Shapiro conditions. After Solonnikov's study about the maximal regularity, to obtain the maximal L_p regularity result in the model problem, Moglievskii [10,11], Mucha and Zajaczkowski [12] and Solonnikov [29] used the Marcinkiewicz–Mikhlin–Lizorkin multiplier theorems together with some Hardy type inequality. Prüss and Simonett [15,16] used $\mathcal{H}^\infty$ calculus and Shibata–Shimizu [25] used the $\mathcal{R}$-boundedness and the Weis operator valued Fourier multiplier theorem.

On the other hand, Denk, Hieber and Prüss [5], Shibata [21], Enomoto and Shibata [6], Enomoto, von Below and Shibata [7], Dario and Shibata [8], Murata [13] used another methods, namely they construct the $\mathcal{R}$ bounded solution operator to the resolvent problem and used the Weis operator valued Fourier multiplier theorem to obtain the maximal L_p in time and L_q in space regularity. In this paper, we follow Enomoto, von Below, and Shibata [6,7] to prove the maximal regularity result for problem (1.10) with help of the $\mathcal{R}$ bounded operator for the generalized resolvent problem:

$$\begin{cases} \lambda\rho + \gamma_1 \mathrm{div}\,\mathbf{u} = f & \text{in } \Omega, \\ \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{T}(\mathbf{u},\gamma_3\rho) = \delta_1 \mathrm{Div}\,\tau + \mathbf{g} & \text{in } \Omega, \\ \lambda\tau + \delta_2\tau - g_\alpha(\nabla\mathbf{u},\tau_1) = \delta_3\mathbf{D}(\mathbf{u}) + \mathbf{h} & \text{in } \Omega, \\ (\mathbf{T}(\mathbf{u},\gamma_3\rho) + \delta_1\tau)\mathbf{n} = \mathbf{k} & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases} \tag{1.12}$$

1.1. *Notation and the definition of uniform domains*

Before stating our main result, we introduce the notation used throughout the paper, and some definitions. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y, and Hol $(U,\mathcal{L}(X,Y))$ the set of all $\mathcal{L}(X,Y)$ valued holomorphic functions defined on a complex domain U. For any domain D and $1 \le p,q \le \infty$, $L_q(D)$, $W^m_q(D)$ and $B^s_{p,q}(D)$ denote the usual Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W^m_q(D)}$ and $\|\cdot\|_{B^s_{q,p}(D)}$ denote their norms, respectively. We set $W^0_q(D) = L_q(D)$, $W^s_q(D) = B^s_{q,q}(D)$ and

$$W^{m,\ell}_q(D) = \{(f,\mathbf{g},\mathbf{h}) \mid f \in W^m_q(D),\ \mathbf{g} \in W^\ell_q(D)^N,\ \mathbf{h} \in W^m_q(D)^{N^2}\}.$$

$C^\infty_0(D)$ denotes the set all $C^\infty(\mathbb{R}^N)$ functions whose supports are compact and contained in D. We set $(f,g)_D = \int_D f(x)g(x)dx$. $L_p((a,b),X)$ and $W^m_p((a,b),X)$ denote the usual Lebesgue space and Sobolev space of X-valued function defined on an interval (a,b), while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W_p^m((a,b),X)}$ denote their norms, respectively. The d-product space of X is defined by $X^d = \{f = (f_1,\ldots,f_d) \mid f_i \in X\ (i=1,\ldots,d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural numbers, real numbers and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For any multi-index $\kappa = (\kappa_1,\ldots,\kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1+\cdots+\kappa_N$ and $\partial_x^\kappa = \partial_1^{\kappa_1}\cdots\partial_N^{\kappa_N}$ with $x=(x_1,\ldots,x_N)$ and $\partial_j = \partial/\partial x_j$. For scalar function f and N-vector of functions $\mathbf{g}$, we set

$$\nabla f = (\partial_1 f,\ldots,\partial_N f), \qquad \nabla\mathbf{g} = (\partial_i g_j \mid i,j=1,\ldots,N),$$
$$\nabla^2 f = (\partial^\alpha f \mid |\alpha|=2), \qquad \nabla^2\mathbf{g} = (\partial^\alpha g_i \mid |\alpha|=2,\ i=1,\ldots,N).$$

For $\mathbf{a} = (a_1\ldots,a_N)$ and $\mathbf{b} = (b_1,\ldots,b_N)$, we set $\mathbf{a}\cdot\mathbf{b} = \langle\mathbf{a},\mathbf{b}\rangle = \sum_{j=1}^N a_j b_j$. For scalar functions f,g and N-vectors of functions $\mathbf{f}$, $\mathbf{g}$ we set $(f,g) = \int_D f(x)g(x)dx$ and $(\mathbf{f},\mathbf{g})_D = \int_D \mathbf{f}(x)\cdot\mathbf{g}(x)dx$. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a,b,\ldots$. The values of constants C and $C_{a,b,\ldots}$ may change from line to line. We use the bold-face letters to denote N-vector valued function and $N\times N$ matrix of functions. And also, we use the Greek letters to denote mass density as well as elastic tensor.

Next, we introduce a definition.

Definition 1.1. Let $1<r<\infty$ and let Ω be a domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. We say that Ω is a uniform $W_r^{2-1/r}$ domain, if there exists positive constants α,β and K such that for any $x_0 = (x_{01},\ldots,x_{0N}) \in \partial\Omega$ there exist a coordinate number j and a $W_r^{2-1/r}$ function $h(x')$ $(x' = (x_1,\ldots,\hat{x}_j,\ldots,x_N))$ defined on $B'_\alpha(x'_0)$ with $x'_0 = (x_{01},\ldots,\hat{x}_{0j},\ldots,x_{0N})$ and $\|h\|_{W_r^{2-1/r}(B'_\alpha(x'_0))} \le K$ such that

$$\Omega\cap B_\beta(x_0) = \{x\in\mathbb{R}^N \mid x_j > h(x')\ (x'\in B'_\alpha(x'_0))\}\cap B_\beta(x_0)$$
$$\partial\Omega\cap B_\beta(x_0) = \{x\in\mathbb{R}^N \mid x_j = h(x')\ (x'\in B'_\alpha(x'_0))\}\cap B_\beta(x_0). \tag{1.13}$$

Here, $(x_1,\ldots,\hat{x}_j,\ldots,x_N) = (x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N)$, $B'_\alpha(x'_0) = \{x'\in\mathbb{R}^{N-1} \mid |x'-x'_0| < \alpha\}$ and $B_\beta(x_0) = \{x\in\mathbb{R} \mid |x-x_0| < \beta\}$.

1.2. Main results

The following theorem represents the main result of this paper.

Theorem 1.2. *Let $N<q<\infty$, $2<p<\infty$ and $R>0$. Then, there exists a time $T>0$ depending on R such that if the initial data $(\theta_0,\mathbf{u}_0,\tau_0)$ for Eqs. (1.1) satisfy*

$$\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-\frac{1}{p})}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \le R, \tag{1.14}$$

the range condition:

$$\frac{\rho_*}{2} < \rho_* + \theta_0 < 2\rho_*, \tag{1.15}$$

and the compatibility condition:

$$(\mathbf{T}(\mathbf{u}_0, P(\rho_*+\theta_0)) + \beta\tau_0)\mathbf{n} = -P(\rho_*)\mathbf{n} \quad \text{on } \Gamma_1, \qquad \mathbf{u}_0 = 0 \quad \text{on } \Gamma_0, \tag{1.16}$$

then problem (1.8) admits a unique solution $(\theta,\mathbf{v},\pi)$ with

$$\theta\in W_p^1((0,T),W_q^1(\Omega)), \qquad \mathbf{v}\in W_p^1((0,T),L_q(\Omega))\cap L_p((0,T),W_q^2(\Omega)), \qquad \pi\in W_p^1((0,T),W_q^1(\Omega))$$

satisfying the conditions:

$$\frac{\rho_*}{4} < \rho_*+\theta_0 < 4\rho_*, \qquad \max_{i,j=1,\ldots,n}\int_0^T \|(\partial u_i/\partial\xi_j)(\cdot,s)ds\|_{L_\infty(\Omega)} < \sigma,$$

and the estimate:

$$\|\theta\|_{W^1_p((0,T),W^1_q(\Omega))} + \|\mathbf{v}\|_{W^1_p((0,T),L_q(\Omega))} + \|\mathbf{v}\|_{L_p((0,T),W^2_q(\Omega))} + \|\pi\|_{W^1_p((0,T),W^1_q(\Omega))} \le CR$$

with some constant C independent of R.

Using the argumentation due to Ströhmer [32], we see that the map $x = \mathbf{X}_{\mathbf{v}}(\xi, t)$ is a diffeomorphism with suitable regularity, so that for problem (1.1) by Theorem 1.2 we have

Theorem 1.3. *Let $N < q < \infty$, $2 < p < \infty$ and $R > 0$. Then, there exists a time $T_1 > 0$ depending on R such that if the initial data $(\theta_0, \mathbf{u}_0, \tau_0)$ for problem (1.1) satisfies the same condition as in Theorem 1.2, then problem (1.1) admits a unique solution $(\rho, \mathbf{u}, \tau)$ with*

$$\rho - \rho_* \in W^1_p((0,T), L_q(\Omega_t)) \cap L_p((0,T), W^1_q(\Omega_t)), \quad \mathbf{u} \in W^1_p((0,T), L_q(\Omega_t)) \cap L_p((0,T), W^2_q(\Omega_t)),$$
$$\tau \in W^1_p((0,T), L^1_q(\Omega_t)) \cap L_p((0,T), W^1_q(\Omega_t)).$$

Remark 1.4. In Theorem 1.3, $v \in W^\ell_p((0,T), W^m_q(\Omega_t))$ means that $\partial^j_t v \in W^m_q(\Omega_t)$ for $t \in (0,T)$ and $j = 0, 1, \ldots, \ell$, where $W^0_p = L_p$, $W^0_q = L_q$ and $\partial^0 v = v$, and

$$\|v\|_{W^\ell_p((0,T),W^m_q(\Omega_t))} = \sum_{j=0}^{\ell} \Big(\int_0^T (\|\partial^j_t v(\cdot,t)\|_{W^m_q(\Omega_t)})^p \, ds \Big)^{1/p} < \infty.$$

Including this introduction, we organize the paper as follows. In Section 2, we discuss the extension of the unit outer normal $\mathbf{n}$ to the whole space, give some proposition about uniform $W^{2-1/r}_r$ domains and prepare some calculus lemmas for the latter use. In Section 3, we show the existence of $\mathcal{R}$-bounded solution operator to problem (1.12) and (1.10). In Section 4, we state the existence of $\mathcal{R}$-bounded solution operator for problem (1.12) and we prove the maximal regularity result for problem. In Section 5, we prove Theorem 1.2.

2. Some properties of the uniform $W^{2-1/r}_r$ domain

In this section, we discuss some properties of the uniform $W^{2-1/r}_r$ domain and we prepare some calculus lemmas for the latter use. Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. We assume that $\nabla\Phi$ and $\nabla\Phi^{-1}$ have the forms: $\nabla\Phi = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(x)$, where $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_{-1}(x)$ are matrices of functions in $W^2_r(\mathbb{R}^N)$ with $N < r < \infty$ such that

$$\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_1, \qquad \|\nabla(B, B_{-1})\|_{L_r(\mathbb{R}^N)} \le M_2. \tag{2.1}$$

Let $\mathcal{A}_{ij}$, $\mathcal{A}_{-ij}$, $B_{ij}(x)$ and $B_{-1ij}(\xi)$ be the (i,j) elements of $\mathcal{A}$, $\mathcal{A}_{-}$, $B(x)$ and $B_{-1}(x)$, respectively. We will choose M_1 small enough eventually, so that in the sequel, we may assume that $0 < M_1 \le 1 \le M_2$. Let $\Omega_+ = \Phi(\mathbb{R}^N_+)$ and $\Gamma_+ = \Phi(\mathbb{R}^N_0)$, where

$$\mathbb{R}^N_+ = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}, \qquad \mathbb{R}^N_0 = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}.$$

The Γ_+ is the boundary of Ω_+ and represented by $\xi = \Phi(x', 0)$ with $x' = (x_1, \ldots, x_{N-1})$. Let

$$N_i = \det \begin{pmatrix} \partial_1\xi_1 & \cdots & \partial_{N-1}\xi_1 \\ \vdots & \cdots & \vdots \\ \partial_1\xi_{i-1} & \cdots & \partial_{N-1}\xi_{i-1} \\ \partial_1\xi_{i+1} & \cdots & \partial_{N-1}\xi_{i+1} \\ \vdots & \cdots & \vdots \\ \partial_1\xi_N & \cdots & \partial_{N-1}\xi_N \end{pmatrix} \quad \text{with } \partial_i\xi_j = \frac{\partial\Phi_j(x)}{\partial x_i}, \tag{2.2}$$

where $\Phi = (\Phi_1, \ldots, \Phi_N)$, let $\tilde{n}_{+i} = (-1)^{N+i} N_i / \sqrt{\sum_{k=1}^N N_k^2}$, let $n_{+i} = \tilde{n}_{+i} \circ \Phi^{-1}$, and let $\mathbf{n}_{\Gamma_+} = (n_{+1}, \ldots, n_{+N})$. We see that $\mathbf{n}_{\Gamma_+}|_{\Gamma_+}$ is the unit outer normal to Γ_+. Moreover, $\mathbf{n}_{\Gamma_+}$ is defined on $\mathbb{R}^N$ and by (2.1)

$$\|\mathbf{n}_{\Gamma_+}\|_{L_\infty(\mathbb{R}^N)} \le C_N, \qquad \|\nabla \mathbf{n}_{\Gamma_+}\|_{W_q^1(\mathbb{R}^N)} \le C_{M_2}. \tag{2.3}$$

Several properties of uniform $W_r^{2-1/r}$ domains are given in the following proposition which was proved in Enomoto and Shibata [6, Proposition 6.1].

Proposition 2.1. *Let $N < r < \infty$ and let Ω be a uniform $W_r^{2-1/r}$ domain in $\mathbb{R}^N$. Let M_1 be any small number $\in (0,1)$. Then, there exist constants $M_2 > 0$, $0 < d^0, d^1, d^2 < 1$, an open set U, at most countably many N-vector of functions Φ_j^0 and Φ_j^1, and points $x_j^0 \in \Gamma_0$, $x_j^1 \in \Gamma_1$ and $x_j^2 \in \Omega$ such that the following assertions hold:*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi_j^i(x) \in \mathbb{R}^N$ $(i = 0, 1)$ are bijective of C^1 class.*
(ii) $\Omega = \left(\bigcup_{i=0}^1 \bigcup_{j=1}^\infty (\Phi_j^i(\mathbb{R}_+^N) \cap B_{d^i}(x_j^i))\right) \cup \left(\bigcup_{j=1}^\infty B_{d^2}(x_j^2)\right)$, $B_{d^2}(x_j^2) \subset \Omega$, $\Phi_j(\mathbb{R}_+^N) \cap B_{d^i}(x_j^i) = \Omega \cap B_{d^i}(x_j^i)$ $(i = 0, 1)$, $\Phi_j^i(\mathbb{R}_0^N) \cap B_{d^i}(x_j^i) = \Gamma_i \cap B_{d^i}(x_j^i)$ $(i = 0, 1)$.
(iii) *There exist C^∞ functions ζ_j^i and $\tilde{\zeta}_j^i (i = 0, 1, 2,\ j = 1, 2, 3, \ldots)$ such that*

$$0 \le \zeta_j^i, \quad \tilde{\zeta}_j^i \le 1, \quad \operatorname{supp} \zeta_j^i,\ \operatorname{supp} \tilde{\zeta}_j^i \subset B_{d^i}(x_j^i), \quad \|\zeta_j^i\|_{W_\infty^2(\mathbb{R}^N)}, \|\tilde{\zeta}_j^i\|_{W_\infty^2(\mathbb{R}^N)} \le c_0,$$
$$\tilde{\zeta}_j^i = 1 \quad on\ \operatorname{supp} \zeta_j^i, \quad \sum_{i=0}^2 \sum_{j=1}^\infty \zeta_j^i = 1 \quad on\ \overline{\Omega}, \quad \sum_{j=1}^\infty \zeta_j^i = 1 \quad on\ \Gamma_i\ (i = 0, 1).$$

Here, c_0 is a constant which depends on M_2, N, q and r, but is independent of $j = 1, 2, 3, \ldots$.
(iv) $\nabla \Phi_j^i = \mathcal{A}_j^i + B_j^i$, $\nabla (\Phi_j^i)^{-1} = \mathcal{A}_{j,-}^i + B_{j,-}^i$, *where $\mathcal{A}_j^i$ and $\mathcal{A}_{j,-}^i$ are $N \times N$ constant orthonormal matrices, and B_j^i and $B_{j,-}^i$ are $N \times N$ matrices of $W_r^{1+i}(\mathbb{R}^N)$ functions defined on $\mathbb{R}^N$ which satisfy the conditions: $\|B_j^i\|_{L_\infty(\mathbb{R}^N)} \le M_1$, $\|B_{j,-}^i\|_{L_\infty(\mathbb{R}^N)} \le M_1$, $\|\nabla B_j^i\|_{L_r(\mathbb{R}^N)} \le M_2$ and $\|\nabla B_{j,-}^i\|_{L_r(\mathbb{R}^N)} \le M_2$ for $i = 0, 1$ and $j = 1, 2, 3, \ldots$.*
(v) *There exists a natural number $L \ge 2$ such that any $L + 1$ distinct sets of $\{B_{d^i(x_j^i)} \mid i = 0, 1, 2,\ j = 1, 2, 3, \ldots\}$ have an empty intersection.*

By Proposition 2.1(v), we have

$$C_q^1 \|f\|_{L_q(\Omega)}^q \le \sum_{i=0}^2 \sum_{j=1}^\infty \|\zeta_j^i f\|_{L_q(\Omega)}^q \le \sum_{i=0}^2 \sum_{j=1}^\infty \|f\|_{L_q(\Omega \cap B_j^i)}^q \le C_q^2 \|f\|_{L_q(\Omega)}^q \tag{2.4}$$

for any $f \in L_q(\Omega)$ and $1 \le q < \infty$ with some positive constants C_q^1 and C_q^2.

In the sequel, we write $B_j^i = B_{d^i}(x_j^i)$, $(\Phi_j^i)^{-1} = \Psi_j^i$, $\Omega_j^1 = \Phi_j^1(\mathbb{R}_+^N)$, and $\Gamma_j^1 = \Phi_j^1(\mathbb{R}_0^N)$ for the sake of simplicity. The Γ_j^1 is the boundary of Ω_j^1. We introduce some properties of the unit outer normal $\mathbf{n}$ to Γ_1, the extension operator $\mathbf{E}$, the space $\mathbf{W}_q^{-1}(\Omega)$ and its norm $\|\cdot\|_{\mathbf{W}_q^{-1}(\Omega)}$, and we prove some inequalities for the later use. From the consideration at the beginning of this section it follows the existence of $\mathbf{n}_k^1 \in W_{r,\mathrm{loc}}^1(\mathbb{R}^N)$ such that $\mathbf{n}_k^1 = \mathbf{n}$ on $\Gamma_1 \cap B_k^1$ and

$$\|\mathbf{n}_k^1\|_{W_r^1(B_k^1)} \le C. \tag{2.5}$$

Let $\tilde{\mathbf{n}} = \sum_{k=1}^\infty \zeta_k^1 \mathbf{n}_k^1$ and $\mathcal{S} = \cup_{k=1}^\infty \operatorname{supp} \zeta_k^1$, and then $\mathbf{n} = \tilde{\mathbf{n}}$ on Γ_1 and $\operatorname{supp} \tilde{\mathbf{n}} \subset \mathcal{S}$. For the notational simplicity, hereinafter we write $\tilde{\mathbf{n}} = \sum_{k=1}^\infty \zeta_k^1 \mathbf{n}_k^1$. Since $\tilde{\mathbf{n}} = \mathbf{n}$ on Γ_1, we write $\mathbf{n} = \tilde{\mathbf{n}}$ unless confusion may occur.

Next, let p_j $(j = 1,2,3,4)$ be numbers such that $\sum_{j=1}^{4}(-j)^k p_j = 1$ for $k = -1,0,1,2$. Given function $f \in L_{1,\mathrm{loc}}(\mathbb{R}^N_+)$, let

$$\iota[f](x) = \begin{cases} f(x', x_N) & (x_N > 0), \\ \displaystyle\sum_{j=1}^{4} p_j f(x', -jx_N) & (x_N < 0). \end{cases}$$

Obviously, $\partial_N^k \iota[f]_{x_N=0+} = \partial_N^k \iota[f]_{x_N=0-} = (\partial_N f)(x', 0+)$, so that $\|\iota[f]\|_{W_q^k(\mathbb{R}^N)} \le C\|f\|_{W_q^k(\mathbb{R}^N_+)}$ for $k = 0,1,2$, where $W_q^0 = L_q$. Moreover, $\iota[\partial_N f] = \partial_N(\sum_{k=1}^{4}(-j)^{-1}p_j f(x', -jx_N))$ for $x_N < 0$ and $\sum_{k=1}^{4}(-j)^{-1}p_j f(x', -jx_N)|_{x_N=0-} = f(x', 0+)$, so that $\|\iota[\partial_N f]\|_{W_q^{-1}(\mathbb{R}^N)} \le C\|f\|_{L_q(\Omega)}$, where $W_q^{-1}(\mathbb{R}^N)$ is the dual space of $W_{q'}^1(\mathbb{R}^N)$.

Let the extension operator $\mathbf{E}$ be defined by

$$\mathbf{E}[f] = \sum_{i=0}^{1}\sum_{j=1}^{\infty} \iota[(\zeta_j^i f)\circ \Phi_j^i]\circ \Psi_j^i + \sum_{j=1}^{\infty}\zeta_j^2 f.$$

For the product fg, $\mathbf{E}[fg]$ is defined by $\mathbf{E}[fg] = \mathbf{E}[f]\mathbf{E}[g]$, and if g is defined on $\mathbb{R}^N$, $\mathbf{E}[fg]$ is defined by $\mathbf{E}[fg] = \mathbf{E}[f]g$. Obviously, $\mathbf{E}[f] = f$ in Ω. Moreover, we have

$$\begin{aligned} \|\mathbf{E}[u]\|_{W_q^k(\mathbb{R}^N)} &\le C\|u\|_{W_q^k(\Omega)} \quad \text{for } k = 0,1,2, \\ \|\mathbf{E}[\nabla u]\|_{W_q^{-1}(\mathbb{R}^N)} &\le C\|u\|_{L_q(\Omega)}. \end{aligned} \tag{2.6}$$

Let

$$\mathbf{W}_q^{-1}(\Omega) = \{f \in L_{1,\mathrm{loc}}(\Omega) \mid \mathbf{E}[f] \in W_q^{-1}(\mathbb{R}^N)\}, \qquad \|f\|_{\mathbf{W}_q^{-1}(\Omega)} = \|\mathbf{E}[f]\|_{W_q^{-1}(\mathbb{R}^N)}.$$

For the later use, we prove

Lemma 2.2. *Let $1 < q < \infty$ and $N < s < \infty$. Assume that $\max(q, q') \le s$. Then, the following assertions hold.*

(1)

$$\|fg\|_{W_q^1(\Omega)} \le C\|f\|_{W_q^1(\Omega)}\|g\|_{W_s^1(\Omega)}, \qquad \|g\|_{L_\infty(\Omega)} \le C\|g\|_{W_s^1(\Omega)}.$$

(2)

$$\begin{aligned} \|\nabla u\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C\|u\|_{L_q(\Omega)}, \\ \|uv\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C_q\|u\|_{\mathbf{W}_q^{-1}(\Omega)}\|v\|_{W_s^1(\Omega)}, \\ \|uv\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C_q\|u\|_{L_q(\Omega)}\|v\|_{L_s(\Omega)}. \end{aligned}$$

(3) *Let g_k $(k = 1,2,\ldots)$ be functions in $W_{s,\mathrm{loc}}^1(\mathbb{R}^N)$ such that*

$$\operatorname{supp} g_k \subset B_k^1, \qquad \|g_k\|_{W_s^1(B_k^1)} \le \gamma_0, \tag{2.7}$$

for some constant γ_0 independent of $k = 1,2,3,\ldots$. Then,

$$\begin{aligned} \Big\|\sum_{k=1}^{\infty}\zeta_k^1 f g_k\Big\|_{\mathbf{W}_q^{-1}(\Omega)} &\le C_q\gamma_0\|f\|_{\mathbf{W}_q^{-1}(\Omega)}, \\ \Big\|\sum_{k=1}^{\infty}\zeta_k^1 f g_k\Big\|_{W_q^k(\Omega)} &\le C_q\gamma_0\|f\|_{W_q^k(\Omega)} \quad (k = 0,1). \end{aligned}$$

Proof. (1) It follows from the Sobolev imbedding theorem that $\|g\|_{L_\infty(\Omega)} \leq C\|g\|_{W^1_s(\Omega)}$, so that we also have $\|(f, \nabla f)g\|_{L_q(\omega)} \leq C\|f\|_{W^1_q(\Omega)}\|g\|_{W^1_s(\Omega)}$. By the Sobolev imbedding theorem, we have

$$\|fg\|_{L_a(\Omega)} \leq C\|f\|_{L_s(\Omega)}\|g\|_{W^1_a(\Omega)} \quad (a = q, q'). \tag{2.8}$$

In fact, by the Hölder inequality, we have $\|fg\|_{L_a(\Omega)} \leq C\|f\|_{L_s(\Omega)}\|g\|_{L_b(\Omega)}$ with $1/a = 1/s + 1/b$. Note that $a \leq s$. If $a = s$, then $b = \infty$ and $N < a < \infty$, so that by the Sobolev imbedding theorem $\|g\|_{L_b(\Omega)} \leq C\|g\|_{W^1_a(\Omega)}$. If $a < s$, then $N(1/a - 1/b) = N/s < 1$, so that by the Sobolev imbedding theorem we also have $\|g\|_{L_b(\Omega)} \leq C\|g\|_{W^1_a(\Omega)}$. Thus, we have (2.8).

Applying (2.8), we have $\|f\nabla g\|_{L_q(\Omega)} \leq C\|f\|_{W^1_q(\Omega)}\|\nabla g\|_{L_s(\Omega)}$. Summing up, we have shown the assertion (1).

(2) The first inequality follows from (2.6). To prove the second one, we observe that

$$|(\mathbf{E}[uv], \varphi)_{\mathbb{R}^N}| \leq \|u\|_{\mathbf{W}^{-1}_q(\Omega)}\|\mathbf{E}[v]\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$. By (2.8) we have

$$\|(\nabla\mathbf{E}[v])\varphi\|_{L_{q'}(\mathbb{R}^N)} \leq C\|\nabla\mathbf{E}[v]\|_{L_s(\mathbb{R}^N)}\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)}. \tag{2.9}$$

Thus, we have $\|\mathbf{E}[v]\varphi\|_{W^1_{q'}(\mathbb{R}^N)} \leq C\|\mathbf{E}[v]\|_{W^1_s(\mathbb{R}^N)}\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$, which implies the second inequality. Analogously, using Hölder's inequality and replacing $\nabla\mathbf{E}[v]$ by $\mathbf{E}[v]$ in (2.9), we have

$$|(\mathbf{E}[uv], \varphi)_{\mathbb{R}^N}| \leq \|\mathbf{E}[u]\|_{L_q(\mathbb{R}^N)}\|\mathbf{E}[v]\varphi\|_{L_{q'}(\mathbb{R}^N)} \leq C\|\mathbf{E}[u]\|_{L_q(\mathbb{R}^N)}\|\mathbf{E}[v]\|_{L_s(\mathbb{R}^N)}\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)},$$

which implies the last inequality.

(3) To prove the first inequality, setting $g = \sum_{k=1}^{\infty} \zeta^1_k g_k$, we observe that

$$|(\mathbf{E}[fg], \varphi)_{\mathbb{R}^N}| = |(\mathbf{E}[f], g\varphi)_{\mathbb{R}^N}| \leq \|f\|_{\mathbf{W}^{-1}_q(\Omega)}\|g\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$. By (2.4) replacing Ω by $\mathbb{R}^N$, (2.7) and (2.9), we have

$$\begin{aligned}
\|\nabla(g\varphi)\|^{q'}_{L_{q'}(\mathbb{R}^N)} &\leq C_{N,q'}\sum_{k=1}^{\infty}\|\nabla(\zeta^1_k g_k\tilde{\zeta}^1_k\varphi)\|^{q'}_{L_{q'}(\mathbb{R}^N)}\\
&\leq C_{N,q'}\sum_{k=1}^{\infty}(\|\nabla(\zeta^1_k g_k)\|^{q'}_{L_s(\mathbb{R}^N)}\|\tilde{\zeta}^1_k\varphi\|^{q'}_{W^1_{q'}(\mathbb{R}^N)} + \|\zeta^1_k g_k\|^{q'}_{L_\infty(\mathbb{R}^N)}\|\tilde{\zeta}^1_k\varphi\|^{q'}_{W^1_{q'}(\mathbb{R}^N)})\\
&\leq C_{N,q'}\gamma_0^{q'}\sum_{k=1}^{\infty}\|\varphi\|^{q'}_{W^1_{q'}(B^1_k)} \leq C_{N,q'}\gamma_0^{q'}\|\varphi\|^{q'}_{W^1_{q'}(\mathbb{R}^N)}.
\end{aligned}$$

Analogously, we also have $\|g\varphi\|_{L_{q'}(\mathbb{R}^N)} \leq C_{N,q'}\gamma_0\|\varphi\|_{W^1_{q'}(\mathbb{R}^N)}$. Thus, we have the first inequality.

Analogously, by (2.4) we easily have the second inequalities, which complete the proof of Lemma 2.2. □

For example, using (2.5) and Lemma 2.2 we have

$$\begin{aligned}
&\|f\mathbf{n}\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}, \qquad \|f\mathbf{n}\|_{W^1_q(\Omega)} \leq C\|f\|_{W^1_q(\Omega)},\\
&\|fg\mathbf{n}\|_{\mathbf{W}^{-1}_q(\Omega)} \leq C\|f\|_{W^{-1}_q(\Omega)}\|g\|_{W^1_q(\Omega)}, \qquad \|fg\mathbf{n}\|_{\mathbf{W}^{-1}_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}\|g\|_{L_q(\Omega)}
\end{aligned} \tag{2.10}$$

with some constant $C > 0$.

3. $\mathcal{R}$ bounded solution operators

In this section, we prove the existence of $\mathcal{R}$ bounded solution operator associated with generalized resolvent problem (1.12). First of all, we introduce the definition of the $\mathcal{R}$ bounded operator family.

Definition 3.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, we have the inequality:

$$\left\{\int_0^1 \left\|\sum_{j=1}^n r_j(u)T_j x_j\right\|_Y^p du\right\}^{1/p} \le C\left\{\int_0^1 \left\|\sum_{j=1}^n r_j x_j\right\|_X^p du\right\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

The resolvent parameter λ in problem (1.12) varies in $\Sigma_{\epsilon,\lambda_0}$ with

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \pi - \epsilon,\ |\lambda| \ge \lambda_0\} \quad (\epsilon \in (0, \pi/2), \lambda_0 > 0).$$

The main result for the $\mathcal{R}$ bounded solution operator is the following theorem.

Theorem 3.2. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \ge \max(q, q')$. Let Ω be a uniform $W_r^{2-1/r}$ domain and $\lambda \in \Sigma_{\epsilon,\lambda_0}$. Set*

$$X_q(\Omega) = \{(f, \mathbf{g}, \mathbf{h}, \mathbf{k}) \mid (f, \mathbf{g}, \mathbf{h}) \in W_q^{1,0}(\Omega), \mathbf{k} \in W_q^1(\Omega)^N\},$$
$$\mathcal{X}_q(\Omega) = \{(F_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4, \mathbf{F}_5) \mid F_1 \in W_q^1(\Omega), \mathbf{F}_2 \in L_q(\Omega)^N, \mathbf{F}_3 \in L_q(\Omega)^N, \mathbf{F}_4 \in L_q(\Omega)^{N^2}, \mathbf{F}_5 \in W_q^1(\Omega)^{N^2}\}.$$

Then, there exists a $\lambda_0 \ge 1$ and an operator family $R(\lambda)$ with

$$R(\lambda) \in \mathrm{Hol}(\Lambda_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,2}(\Omega)))$$

such that for any $(f, \mathbf{g}, \mathbf{h}, \mathbf{k}) \in X_q(\Omega)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $(\rho, \mathbf{u}, \tau) = R(\lambda)(f, \mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k}, \mathbf{h})$ is a unique solution to problem (1.12).

Moreover, there exists a constant C such that

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau\partial\tau)^\ell(\lambda R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), W_q^{1,0}(\Omega))}(\{(\tau\partial\tau)^\ell(\gamma R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^2})}(\{(\tau\partial\tau)^\ell(\lambda^{1/2}\nabla P_v R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le C \quad (\ell = 0, 1),\\
\mathcal{R}_{\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{N^3})}(\{(\tau\partial\tau)^\ell(\nabla^2 P_v R(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) &\le C \quad (\ell = 0, 1),
\end{aligned} \tag{3.1}$$

with $\lambda = \gamma + i\tau$. Here, P_v is the projection operator defined by $P_v(\rho, \mathbf{u}, \tau) = \mathbf{u}$.

Remark 3.3. The F_1, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ and $\mathbf{F}_5$ are variables corresponding to f, $\mathbf{g}$, $\lambda^{1/2}\mathbf{k}$, $\nabla\mathbf{k}$, and $\mathbf{h}$, respectively.

In the sequel, we prove Theorem 3.2. To prove Theorem 3.2, we reduce the problem to the Lamé equation:

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) = \mathbf{g} & \text{in } \Omega,\\ \mathbf{S}(\mathbf{u})\mathbf{n} = \mathbf{k} & \text{on } \Gamma_1,\\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases} \tag{3.2}$$

According to Enomoto, von Below and Shibata [7], we know

Theorem 3.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $N < r < \infty$. Assume that $r \geq \max(q, q')$. Let Ω be a uniform $W_r^{2-1/r}$ domain. Let*

$$Y_q(\Omega) = \{(\mathbf{g}, \mathbf{k}) \mid \mathbf{g} \in L_q(\Omega)^N, \mathbf{k} \in W_q^1(\Omega)^N\},$$
$$\mathcal{Y}_q(\Omega) = \{(\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \mid \mathbf{F}_2 \in L_q(\Omega)^N, \mathbf{F}_3 \in L_q(\Omega)^N, \mathbf{F}_4 \in L_q(\Omega)^{N^2}\}.$$

Then there exist a $\lambda_0 \geq 1$ and an operator family $\mathcal{A}(\lambda)$ with

$$\mathcal{A}(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}_q(\Omega), W_q^2(\Omega)^N))$$

such that for any $(\mathbf{g}, \mathbf{k}) \in Y_q(\Omega)$ and $\lambda \in \Lambda_{\epsilon,\lambda_0}$, $\mathbf{u} = \mathcal{A}(\lambda)(\mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k})$ is a unique solution of problem (3.2) and $\mathcal{A}(\lambda)$ satisfy the estimates

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), L_q(\Omega)^{\tilde{N}})}(\{(\tau\partial_\tau)^\ell (G_\lambda \mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq C \quad (\ell = 0, 1)$$

with $\lambda = \gamma + i\tau$, where we set $\tilde{N} = 2N + N^2 + N^3$ and $G_\lambda \mathbf{u} = (\lambda\mathbf{u}, \gamma\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u})$.

Setting $\theta = \lambda^{-1}(f - \gamma_1 \mathrm{div}\,\mathbf{u})$ and $\tau = (\lambda + \delta_2)^{-1}(\delta_3 D(\mathbf{u}) + g_\alpha(\nabla\mathbf{u}, \tau_1) + \mathbf{h})$ for the case $\lambda \neq 0$ in (1.12), we have

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) = \mathbf{g} - \lambda^{-1}\nabla(\gamma_3 f) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,\mathbf{h} & \\ \qquad + \lambda^{-1}\nabla(\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u}) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})) & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n} = \mathbf{k} + (\lambda^{-1}\gamma_3 f - \delta_1(\lambda+\delta_2)^{-1}\mathbf{h})\mathbf{n} & \\ \qquad - (\lambda^{-1}\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u} + \delta_1(\lambda+\delta_2)^{-1}(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})))\mathbf{n} & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0. \end{cases}$$

Thus, $\mathbf{g} - \lambda^{-1}\nabla(\gamma_3 f) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,\mathbf{h}$ and $\mathbf{k} + (\lambda^{-1}\gamma_3 f - \delta_1(\lambda+\delta_2)^{-1}\mathbf{h})\mathbf{n}$ being renamed $\mathbf{g}$ and $\mathbf{k}$, respectively, for the sake of simplicity, we consider the following equations:

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u}) - B_1(\lambda)(\mathbf{u}) = \mathbf{g} & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n} - B_2(\lambda)(\mathbf{u}) = \mathbf{k} & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \end{cases} \tag{3.3}$$

where we have set

$$\begin{aligned} B_1(\lambda)(\mathbf{u}) &= \lambda^{-1}\nabla(\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u}) + \delta_1(\lambda+\delta_2)^{-1}\mathrm{Div}\,(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})), \\ B_2(\lambda)(\mathbf{u}) &= -(\lambda^{-1}\gamma_1\gamma_3 \mathrm{div}\,\mathbf{u} + \delta_1(\lambda+\delta_2)^{-1}(g_\alpha(\nabla\mathbf{u}, \tau_1) + \delta_3\mathbf{D}(\mathbf{u})))\mathbf{n}. \end{aligned} \tag{3.4}$$

To prove Theorem 3.2, we use the following two lemmas about the $\mathcal{R}$-norms.

Lemma 3.5 ([5]). *Let X, Y and Z be Banach space and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families*

1. *If X and Y are Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$, then $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

2. *If X, Y and Z are Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively, then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{T}\mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

Lemma 3.6 ([2]). *Let $1 < p, q < \infty$ and let D be a domain in $\mathbb{R}^N$.*

1. *Let $m(\lambda)$ be a bounded function defined on a subset Λ in a complex plane $\mathbb{C}$ and let $M_m(\lambda)$ be a multiplication operator with $m(\lambda)$ defined by $M_m(\lambda)f = m(\lambda)f$ for any $f \in L_q(D)$.*

Then

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \leq C_{n,q,D}\|m\|_{L_\infty}.$$

2. *Let $n(\tau)$ be a C^1 function defined on $\mathbb{R}\setminus\{0\}$ that satisfies the conditions: $|n(\tau)| \leq \gamma$ and $|\tau n'(\tau)| \leq \gamma$ with some constants $\gamma > 0$ for any $\gamma \in \mathbb{R}\setminus\{0\}$. Let T_n be an operator valued Fourier multiplier defined by $T_n f = \mathcal{F}^{-1}[n\mathcal{F}[f]]$ for any f with $\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)$. Then, T_n is extended to bounded linear operator from $L_q\mathbb{R}, L_q(D)$ into itself. Moreover, denoting this extension also by T_n, we have*

$$\|T_n\|_{\mathcal{L}(L_q(\mathbb{R}, L_q(D)))} \leq C_{p,q,D}\gamma.$$

Hereinafter, we consider problem (3.3). Let $\mathcal{A}(\lambda)$ be the operator given in Theorem 3.4, and let $\mathbf{u} = \mathcal{A}(\lambda)F_\lambda(\mathbf{g},\mathbf{k})$ in (3.3), where $F_\lambda(\mathbf{g},\mathbf{k}) = (\mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k})$. By Theorem 3.4, (3.3) and (3.4), we have

$$\begin{cases} \gamma_2\lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u}) - B_1(\lambda)(\mathbf{u}) = \mathbf{g} - \mathcal{C}_1(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n} - B_2(\lambda)(\mathbf{u}) = \mathbf{k} - \mathcal{C}_2(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) & \text{on } \Gamma_1, \\ \mathbf{u} = 0 & \text{on } \Gamma_0, \end{cases} \tag{3.5}$$

where we have set

$$\begin{aligned} \mathcal{C}_1(\lambda)\mathbf{F} &= \lambda^{-1}\nabla(\gamma_1\gamma_3\operatorname{div}\mathcal{A}(\lambda)\mathbf{F}) + \delta_1(\lambda+\delta_2)^{-1}\operatorname{Div}(g_\alpha(\nabla\mathcal{A}(\lambda)\mathbf{F},\tau_1) + \delta_3\mathbf{D}(\mathcal{A}(\lambda)\mathbf{F})), \\ \mathcal{C}_2(\lambda)\mathbf{F} &= -(\lambda^{-1}\gamma_1\gamma_3\operatorname{div}\mathcal{A}(\lambda)\mathbf{F} + \delta_1(\lambda+\delta_2)^{-1}(g_\alpha(\nabla\mathcal{A}(\lambda)\mathbf{F},\tau_1) + \delta_3\mathbf{D}(\mathcal{A}(\lambda)\mathbf{F})))\mathbf{n}. \end{aligned} \tag{3.6}$$

Let $\mathcal{E}_\lambda\mathbf{u} = (\gamma_2\lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u}) - B_1(\lambda)(\mathbf{u}), \mathbf{S}(\mathbf{u})\mathbf{n} - B_2(\lambda)(\mathbf{u}))$ and $\mathcal{G}_\lambda\mathbf{F} = (C_1(\lambda)\mathbf{F}, C_2(\lambda)\mathbf{F})$. For $\mathbf{F} = (F_1, \mathbf{F}', \mathbf{F}_5) \in \mathcal{X}_q(\Omega)$ with $\mathbf{F}' = (\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \in \mathcal{Y}_q(\Omega)$, we may write Eq. (3.5) in the form:

$$\mathcal{E}_\lambda\mathcal{A}(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) = (\mathbf{I} - \mathcal{G}_\lambda F_\lambda)(\mathbf{g},\mathbf{k}), \tag{3.7}$$

where $\mathbf{I}$ is the identity map from $Y_q(\Omega)$ into itself.

Let λ_1 be any positive number $\geq \lambda_0$. By (1.11), Lemma 2.2(1), (2.10), Lemma 3.5, Lemma 3.6 and Theorem 3.4, we have

$$\begin{aligned} &\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\mathcal{C}_1(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq C\lambda_1^{-1} \quad (\ell = 0,1), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), L_q(\Omega)^N)}(\{(\tau\partial_\tau)^\ell\lambda^{1/2}\mathcal{C}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq C\lambda_1^{-1} \quad (\ell = 0,1), \\ &\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega), L_q(\Omega)^{N^2})}(\{(\tau\partial_\tau)^\ell\nabla\mathcal{C}_2(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq C\lambda_1^{-1} \quad (\ell = 0,1). \end{aligned} \tag{3.8}$$

In fact, for any $n \in \mathbb{N}$, $\lambda_j \in \Sigma_{\epsilon,\lambda_1}$, $\mathbf{F}_j \in \mathcal{Y}_q(\Omega)$, and independent, symmetric, $\{-1,1\}$-valued random variables r_j $(j = 1,\ldots,n)$, we have

$$\begin{aligned} &\int_0^1 \left\|\sum_{j=1}^n r_j(u)\nabla C_2(\lambda_j)\mathbf{F}_j\right\|_{L_q(\Omega)} du \\ &\leq C_{\rho_1}\int_0^1\left(\left\|\sum_{j=1}^n r_j(u)\lambda_j^{-1}\mathcal{A}(\lambda_j)\mathbf{F}_j\right\|_{W_q^2(\Omega)} + \left\|\sum_{j=1}^n r_j(u)(\lambda_j+\delta_2)^{-1}\mathcal{A}(\lambda_j)\mathbf{F}_j\right\|_{W_q^2(\Omega)}\right) du \\ &\leq C_{\rho_1}(\lambda_1^{-1} + (\lambda_1+\delta_2)^{-1})\int_0^1\left\|\sum_{j=1}^n r_j(u)\mathcal{A}(\lambda_j)\mathbf{F}_j\right\|_{W_q^2(\Omega)} du \\ &\leq C_{\rho_1}\lambda_1^{-1}\int_0^1\left\|\sum_{j=1}^n r_j(u)\mathbf{F}_j\right\|_{L_q(\Omega)} du. \end{aligned}$$

Analogously, we have

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u) C_1(\lambda_j)\mathbf{F}_j \right\|_{L_q(\Omega)} du \le C_{\rho_1}(\lambda_1^{-1} + (\lambda_1+\delta_2)^{-1}) \int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathcal{A}(\lambda_j)\mathbf{F}_j \right\|_{W_q^2(\Omega)} du$$
$$\le C_{\rho_1}\lambda_1^{-1} \int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathbf{F}_j \right\|_{L_q(\Omega)} du.$$

And also,

$$\int_0^1 \left\| \sum_{j=1}^n r_j(u)\lambda_j^{1/2} C_2(\lambda_j)\mathbf{F}_j \right\|_{L_q(\Omega)} du \le C_{\rho_1}(\lambda_1^{-1} + (\lambda_1+\delta_2)^{-1}) \int_0^1 \left\| \sum_{j=1}^n r_j(u)\lambda_j^{1/2}\mathcal{A}(\lambda_j)\mathbf{F}_j \right\|_{W_q^1(\Omega)} du$$
$$\le C_{\rho_1}\lambda_1^{-1} \int_0^1 \left\| \sum_{j=1}^n r_j(u)\mathbf{F}_j \right\|_{L_q(\Omega)} du.$$

Thus, we have (3.5) for $\ell = 0$. Analogously, we have (3.5) for $\ell = 1$.

In particular, by (3.8) we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),\mathcal{Y}_q(\Omega))}(\{(\tau\partial_\tau)^\ell F_\lambda \mathcal{G}(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \le C\lambda_1^{-1} \quad (\ell = 0, 1). \tag{3.9}$$

We choose $\lambda_1 \ge \lambda_0$ so large that

$$C\lambda_1^{-1} \le 1/2 \tag{3.10}$$

in (3.9). Let $\|(\mathbf{g},\mathbf{k})\|_{Y_q(\Omega)} = \|\mathbf{g}\|_{L_q(\Omega)} + \|\mathbf{k}\|_{W_q^1(\Omega)}$ and $\|\mathbf{F}\|_{\mathcal{Y}_q(\Omega)} = \sum_{k=2,3,4} \|\mathbf{F}_k\|_{L_q(\Omega)}$. By (3.10)

$$\|F_\lambda[\mathcal{G}_\lambda F_\lambda(\mathbf{g},\mathbf{k})]\|_{\mathcal{Y}_q(\Omega)} = \|F_\lambda \mathcal{G}_\lambda(F_\lambda(\mathbf{g},\mathbf{k}))\|_{\mathcal{Y}_q(\Omega)} \le (1/2)\|F_\lambda(\mathbf{g},\mathbf{k})\|_{\mathcal{Y}_q(\Omega)}.$$

Since $\|F_\lambda(\mathbf{g},\mathbf{k})\|_{\mathcal{Y}_q(\Omega)}$ is equivalent norms to $\|(\mathbf{g},\mathbf{k})\|_{Y_q(\Omega)}$ provided that $\lambda \ne 0$, $\mathbf{I} - \mathcal{G}_\lambda F_\lambda$ has its inverse operator $(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}$ in $Y_q(\Omega)$. By (3.7), $\mathcal{E}_\lambda \mathcal{A}(\lambda)F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}(\mathbf{g},\mathbf{k}) = (\mathbf{g},\mathbf{k})$, so that problem (3.3) admits a solution $\mathbf{u} = \mathcal{A}(\lambda)F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}(\mathbf{g},\mathbf{k})$. The uniqueness follows from the existence of solutions to the dual equations. Moreover, $F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1} = (\mathbf{I} - F_\lambda \mathcal{G}_\lambda)^{-1} F_\lambda$. Thus, if we define the operator $\mathcal{B}(\lambda) = \mathcal{A}(\lambda)(\mathbf{I} - F_\lambda\mathcal{G}_\lambda)^{-1}$, then $\mathbf{u} = \mathcal{B}(\lambda)F_\lambda(\mathbf{g},\mathbf{k}) = \mathcal{A}(\lambda)F_\lambda(\mathbf{I} - \mathcal{G}_\lambda F_\lambda)^{-1}(\mathbf{g},\mathbf{k})$ is a unique solution of problem (3.3), and by Theorem 3.4, Lemma 3.5, (3.9), and (3.10), we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\Omega),L_q(\Omega)^{\tilde N})}(\{(\tau\partial\tau)^\ell(G_\lambda \mathcal{B}(\lambda)) \mid \lambda \in \Lambda_{\epsilon,\lambda_0}\}) \le C \quad (\ell = 0, 1). \tag{3.11}$$

For $\mathbf{F} = (F_1, \mathbf{F}', \mathbf{F}_5) \in \mathcal{X}_q(\Omega)$ with $\mathbf{F}' = (\mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \in \mathcal{Y}_q(\Omega)$, let $R(\lambda)\mathbf{F}$ be defined by

$$R(\lambda)\mathbf{F} = (\lambda^{-1}(F_1 - \gamma_1 \operatorname{div}\mathcal{B}(\lambda)\mathbf{F}')), \mathcal{B}(\lambda)\mathbf{F}', (\lambda+\delta_2)^{-1}(\delta_3\mathbf{D}(\mathcal{B}(\lambda)(\lambda)\mathbf{F}') + g_\alpha(\nabla\mathcal{B}(\lambda)\mathbf{F}', \tau_1) + \mathbf{F}_5),$$

and then by Lemma 3.5 and (3.11), we see that $R(\lambda)$ is the required operator in Theorem 3.2, which completes the proof of Theorem 3.2.

4. L_p–L_q maximal regularity for problem (1.10)

In this section, we shall prove the following theorem concerned with the L_p–L_q maximal regularity.

Theorem 4.1. *Let $1 < p, q < \infty$, $N < r < \infty$, and $\max(q, q') \le r$ ($q' = \frac{q}{q-1}$). Let T be any positive number. Assume that Ω is a uniform $W_r^{2-\frac{1}{r}}$ domain. Let*

$$\rho_0 \in W_q^1(\Omega), \qquad \mathbf{u}_0 \in B_{q,p}^{2(1-1/p)}(\Omega)^N, \qquad \tau_0 \in W_q^1(\Omega)^{N^2}$$

be initial data for problem (1.10), and let

$$f \in L_p((0,T), W_q^1(\Omega)), \qquad \mathbf{g} \in L_p((0,T), L_q(\Omega)), \qquad \mathbf{h} \in L_p((0,T), W_q^1(\Omega)^{N^2}),$$
$$\mathbf{k} \in L_p((0,T), W_q^1(\Omega)^N) \cap W_p^1((0,T), \mathbf{W}_q^{-1}(\Omega)^N),$$

be right members for problem (1.10). Assume that they satisfy the compatibility condition:

$$(\mathbf{T}(\mathbf{u}_0, \gamma_3\rho_0) + \delta_1\tau_0)\mathbf{n} = \mathbf{k}|_{t=0} \quad \text{on } \Gamma_1, \qquad \mathbf{u}_0 = 0 \quad \text{on } \Gamma_0. \tag{4.1}$$

Then, problem (1.10) admits unique solutions ρ, $\mathbf{u}$ and τ with

$$\rho \in W_p^1((0,T), W_q^1(\Omega)), \qquad \mathbf{u} \in L_p((0,T), W_q^2(\Omega)^N) \cap W_p^1((0,T), L_q(\Omega)^N), \qquad \tau \in W_p^1((0,T), W_q^1(\Omega)^{N^2})$$

possessing the estimate:

$$[\![(\rho, \mathbf{u}, \tau)]\!]_t \le Ce^{\gamma t}(\|(\rho_0, \tau_0)\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|(f, \mathbf{g}, \mathbf{h})\|_{L_p((0,t), W_q^{1,0}(\Omega))}$$
$$+ \|\mathbf{k}\|_{L_p((0,t), W_q^1(\Omega))} + \|\partial_t \mathbf{k}\|_{L_p((0,t), \mathbf{W}_q^{-1}(\Omega))}) \tag{4.2}$$

for any $t \in (0,T)$ with some positive constants γ and C, where we have set

$$[\![(\rho, \mathbf{u}, \tau)]\!]_t = \|(\rho, \tau)\|_{W_p^1((0,t), W_q^1(\Omega))} + \|\mathbf{u}\|_{L_p((0,t), W_q^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L_p((0,t), L_q(\Omega))} \tag{4.3}$$

and the constant C in (4.2) depends on ρ_0 and ρ_1.

To prove Theorem 4.1, first of all we transform problem (1.10) to the zero initial data case. To this end, we take a domain Ω_1 such that $\partial\Omega_1 = \Gamma_0$ and $\Omega \subset \Omega_1$. The Ω_1 is a uniform $W_r^{2-1/r}$ $(N < r < \infty)$ domain. Let $\mathbf{u}_0 \in B_{q,p}^{2(1-1/p)}(\Omega)$ be an initial velocity field for problem (1.10) and let $\tilde{\mathbf{u}}_0 = (\tilde{u}_{01}, \dots, \tilde{u}_{0N})$ be an extension of $\mathbf{u}_0$ to Ω_1 such that $\mathbf{u}_0 = \tilde{\mathbf{u}}_0$ on Ω and $\|\tilde{\mathbf{u}}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega_1)} \le C\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$. We consider the time-shifted heat equations:

$$\partial_t v_j + \lambda_0 v_j - \mu\Delta v_j = 0 \quad \text{in } \Omega_1 \times (0,\infty), \qquad v_j|_{\Gamma_0} = 0, \qquad v_j|_{t=0} = \tilde{u}_{0j} \tag{4.4}$$

$(j = 1, \dots, N)$. Since $\tilde{u}_{0j}$ satisfies the compatibility condition: $\tilde{u}_{0j}|_{\Gamma_0} = u_{0j}|_{\Gamma_0} = 0$ as follows from (4.1), employing the similar argumentation to that in Shibata [21,22], we see that there exist v_j $(j = 1, \dots, N)$ such that

$$v_j \in L_p((0,\infty), W_q^2(\Omega_1)) \cap W_p^1((0,\infty), L_q(\Omega_1)),$$
$$\|\partial_t v_j\|_{L_p((0,\infty), L_q(\Omega_1))} + \|v_j\|_{L_p((0,\infty), W_q^2(\Omega_1))} \le C\|\tilde{u}_{0j}\|_{B_{q,p}^{2(1-1/p)}(\Omega_1)} \le C\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}. \tag{4.5}$$

Set $\mathbf{v} = (v_1, \dots, v_N)$. In problem (1.10), we set $\rho = \rho_0 + \theta$, $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\tau = \tau_0 + \omega$, and then θ, $\mathbf{w}$ and ω satisfy the following equations:

$$\begin{cases} \partial_t\theta + \gamma_1 \operatorname{div} \mathbf{w} = f' & \text{in } \Omega \times (0,T), \\ \gamma_2\partial_t \mathbf{w} - \operatorname{Div} \mathbf{T}(\mathbf{w}, \gamma_3\theta) = \delta_1 \operatorname{Div} \omega + \mathbf{g}' & \text{in } \Omega \times (0,T), \\ \partial_t\omega + \delta_2\omega - g_\alpha(\nabla\mathbf{w}, \tau_1) = \delta_3 \mathbf{D}(\mathbf{w}) + \mathbf{h}' & \text{in } \Omega \times (0,T), \\ (\mathbf{T}(\mathbf{w}, \gamma_3\theta) + \delta_1\omega)\mathbf{n} = \mathbf{k}' & \text{on } \Gamma_1 \times (0,T), \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \times (0,T), \\ (\theta, \mathbf{w}, \omega)|_{t=0} = (0,0,0) & \text{in } \Omega, \end{cases} \tag{4.6}$$

with $f' = f - \gamma_1 \operatorname{div} \mathbf{v}$, $\mathbf{g}' = \mathbf{g} - \gamma_2\partial_t \mathbf{v} + \operatorname{Div} \mathbf{T}(\mathbf{v}, \gamma_3\rho_0) + \delta_1 \operatorname{Div} \tau_0$, $\mathbf{h}' = \mathbf{h} - \delta_2\tau_0 + g_\alpha(\nabla\mathbf{v}, \tau_1) + \delta_3\mathbf{D}(\mathbf{v})$, and $\mathbf{k}' = \mathbf{k} - (\mathbf{T}(\mathbf{v}, \gamma_3\rho_0) + \delta_1\tau_0)\mathbf{n}$. By (4.5) and Lemma 2.2(1) with $s = r$, (2.10) and (1.11), we have

$$\|(f', \mathbf{g}', \mathbf{h}')\|_{L_p((0,t), W_q^{1,0}(\Omega))} + \|\mathbf{k}'\|_{L_p((0,t), W_q^1(\Omega))} + \|\partial_t \mathbf{k}'\|_{L_p((0,t), \mathbf{W}_q^{-1}(\Omega))} \le C\mathcal{D}_t \tag{4.7}$$

with

$$\mathcal{D}_t = \|(\rho_0, \tau_0)\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2-1/r}(\Omega)} + \|(f, \mathbf{g}, \mathbf{h})\|_{L_p((0,t), W_q^{1,0}(\Omega))}$$
$$+ \|\mathbf{k}\|_{L_p((0,t), W_q^1(\Omega))} + \|\partial_t \mathbf{k}\|_{L_p((0,t), \mathbf{W}_q^{-1}(\Omega))}.$$

Thus, from now on we consider problem (4.6). We modify the right members to consider the problem on $\mathbb{R}$ for time. Given any function $f(\cdot,t)$ defined on $(0,T)$, let f_0 denote the zero extension of f to $(-\infty,0)$, namely $f_0(\cdot,t)=f(\cdot,t)$ for $t\in(0,T)$ and $f_0(\cdot,t)=0$ for $t\in(-\infty,0)$. Let E_t be an operator defined by

$$[E_tf](\cdot,s)=\begin{cases} f_0(\cdot,s) & \text{for } s<t,\\ f_0(\cdot,2t-s) & \text{for } s>t.\end{cases} \tag{4.8}$$

Obviously, $[E_tf](\cdot,s)=0$ for $s\notin(0,2t)$. Moreover, if $f|_{t=0}=0$, then we have

$$\partial_s[E_tf](\cdot,s)=\begin{cases} 0 & \text{for } s\notin(0,2t),\\ (\partial_sf)(\cdot,s) & \text{for } s\in(0,t),\\ -(\partial_sf)(\cdot,2t-s) & \text{for } s\in(t,2t).\end{cases} \tag{4.9}$$

For $t\in(0,T)$, let

$$F=E_t[f'],\qquad \mathbf{G}=E_t[\mathbf{g}'],\qquad \mathbf{H}=E_t[\mathbf{h}'],\qquad \mathbf{K}=E_t[\mathbf{k}'].$$

By the compatibility condition (4.1), $\mathbf{k}'|_{t=0}=0$, so that by (4.9), we have

$$\begin{aligned} &\partial_s\mathbf{K}=(\partial_s\mathbf{k}')(\cdot,s)\quad \text{for } s\in(0,t),\qquad \partial_s\mathbf{K}=-(\partial_s\mathbf{k}')(\cdot,2t-s)\quad \text{for } s\in(t,2t),\\ &\partial_s\mathbf{K}=0\quad \text{for } s\notin(0,2t).\end{aligned} \tag{4.10}$$

First, we consider the whole time problem:

$$\begin{cases} \partial_t\theta+\gamma_1\operatorname{div}\mathbf{w}=F & \text{in } \Omega\times\mathbb{R}\\ \gamma_2\partial_t\mathbf{w}-\operatorname{Div}\mathbf{T}(\mathbf{w},\gamma_3\theta)=\delta_1\operatorname{Div}\omega+\mathbf{G} & \text{in } \Omega\times\mathbb{R}\\ \partial_t\omega+\delta_2\omega-g_\alpha(\nabla\mathbf{w},\tau_1)=\delta_3\mathbf{D}(\mathbf{w})+\mathbf{H} & \text{in } \Omega\times\mathbb{R}\\ (\mathbf{T}(\mathbf{w},\gamma_3\theta)+\delta_1\omega)\mathbf{n}=\mathbf{K} & \text{on } \Gamma_1\times\mathbb{R},\\ \mathbf{w}=0 & \text{on } \Gamma_0\times\mathbb{R}.\end{cases} \tag{4.11}$$

Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace–Fourier transform and the inverse Laplace–Fourier transform with respect to t defined by

$$\mathcal{L}[f](\lambda)=\hat f=\int_{-\infty}^{\infty}e^{-(\gamma+i\tau)t}f(t)\,dt\quad(\lambda=\gamma+i\tau),\qquad \mathcal{L}^{-1}[g](t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(\gamma+i\tau)t}g(\tau)\,d\tau.$$

Let $\mathcal{F}_t$ and $\mathcal{F}_\tau^{-1}$ be the Fourier transform with respect to t and the inverse Fourier transform with respect to τ defined by

$$\mathcal{F}[f](\tau)=\int_{-\infty}^{\infty}e^{-i\tau t}f(t)\,dt,\qquad \mathcal{F}^{-1}[g](t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau t}g(\tau)\,d\tau.$$

We see that

$$\mathcal{L}[f](\lambda)=\mathcal{F}_t[e^{-\gamma t}f(t)],\qquad \mathcal{L}^{-1}[g](t)=e^{\gamma t}\mathcal{F}_\tau^{-1}[g(\tau)](t). \tag{4.12}$$

Applying the Laplace–Fourier transform to (4.11), we have

$$\begin{cases} \lambda\hat\theta+\gamma_1\operatorname{div}\hat{\mathbf{w}}=\hat F & \text{in } \Omega\\ \gamma_2\lambda\hat{\mathbf{w}}-\operatorname{Div}\mathbf{T}(\hat{\mathbf{w}},\gamma_3\hat\theta)=\delta_1\operatorname{Div}\hat\omega+\hat{\mathbf{G}} & \text{in } \Omega\\ \lambda\hat\omega+\delta_2\hat\omega-g_\alpha(\nabla\hat{\mathbf{w}},\tau_1)=\delta_3\mathbf{D}(\hat{\mathbf{w}})+\hat{\mathbf{H}} & \text{in } \Omega\\ (\mathbf{T}(\hat{\mathbf{w}},\gamma_3\hat\theta)+\delta_1\hat\omega)\mathbf{n}=\hat{\mathbf{K}} & \text{on } \Gamma_1,\\ \hat{\mathbf{w}}=0 & \text{on } \Gamma_0.\end{cases} \tag{4.13}$$

Let $R(\lambda)$ be the solution operator to problem (1.12) given in Theorem 3.2, and then we have

$$(\theta,\mathbf{w},\omega)=\mathcal{L}^{-1}[R(\lambda)(\hat F,\hat{\mathbf{G}},\lambda^{1/2}\hat{\mathbf{K}},\nabla\hat{\mathbf{K}},\hat{\mathbf{H}})]. \tag{4.14}$$

Let $\Lambda_\gamma^{1/2} f$ be the operator defined by

$$\Lambda_\gamma^{1/2} f = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[f](\lambda)].$$

Note that $\lambda^{1/2}\hat{\mathbf{K}} = \mathcal{L}[\Lambda_\gamma^{1/2}\mathbf{K}]$. To estimate $(\theta, \mathbf{w}, \omega)$, we quote the Weis operator valued Fourier multiplier theorem. Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact support and the Schwartz space of rapidly decreasing X valued function, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]] \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)). \tag{4.15}$$

The following theorem is obtained by Weis [33].

Theorem 4.2. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}\left(\left\{\left(\tau\frac{d}{d\tau}\right)^\ell M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\right\}\right) \le \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (4.15) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 4.3. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

Applying the Weis theorem stated above to $(\theta, \mathbf{w}, \omega)$ defined in (4.14), we have

$$\begin{aligned} &\|e^{-\gamma s}(\partial_t\theta, \partial_t\omega)\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \|e^{-\gamma s}(\partial_t\mathbf{w}, \Lambda_\gamma^{1/2}\nabla\mathbf{w}, \nabla^2\mathbf{w})\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad \le C(\|e^{-\gamma s}(F, \mathbf{G}, \mathbf{H})\|_{L_p(\mathbb{R}, W_q^{1,0}(\Omega))} + \|e^{-\gamma s}(\Lambda_\gamma^{1/2}\mathbf{K}, \nabla\mathbf{K})\|_{L_p(\mathbb{R}, L_q(\Omega))}) \end{aligned} \tag{4.16}$$

for any $\gamma \ge \lambda_0 + 1$ with some constants C independent of γ, where λ_0 is the constant given in Theorem 3.2. By using the fact due to Shibata [23, Appendix], Lemmas 2.2 and 3.6, we can prove easily that

$$\begin{aligned} &\|e^{-\gamma s}\Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R}, L_q(\Omega))} \le C\{\|e^{-\gamma s}\partial_s f\|_{L_p(\mathbb{R}, \mathbf{W}_q^{-1}(\Omega))} + \|e^{-\gamma s} f\|_{L_p(\mathbb{R}, W_q^1(\Omega))}\}, \\ &\|e^{-\gamma s}\gamma f\|_{L_p(\mathbb{R}, L_q(\Omega))} \le C\|e^{-\gamma s}\partial_s f\|_{L_p(\mathbb{R}, L_q(\Omega))}, \\ &\|e^{-\gamma s}\partial_t f\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma s} f\|_{L_p(\mathbb{R}, W_q^2(\Omega))} \le C\|e^{-\gamma s}(\partial_s f, \Lambda_\gamma^{1/2}\nabla f, \nabla^2 f)\|_{L_p(\mathbb{R}, L_q(\Omega))}, \end{aligned} \tag{4.17}$$

which, combined with (4.16), furnishes that

$$\begin{aligned} &\gamma\|e^{-\gamma s}(\theta, \mathbf{w}, \omega)\|_{L_p(\mathbb{R}, W_q^{1,0}(\Omega))} + \|e^{-\gamma s}(\partial_t\theta, \partial_t\omega)\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \|e^{-\gamma s}\partial_t\mathbf{w}\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma s}\mathbf{w}\|_{L_p(\mathbb{R}, W_q^2(\Omega))} \\ &\quad \le C(\|e^{-\gamma s}(F, \mathbf{G}, \mathbf{H})\|_{L_p(\mathbb{R}, W_q^{1,0}(\Omega))} + \|e^{-\gamma s}\nabla\mathbf{K}\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma s}\partial_s\mathbf{K}\|_{L_p(\mathbb{R}, \mathbf{W}_q^{-1}(\Omega))}) \end{aligned} \tag{4.18}$$

for any $\gamma \ge \lambda_0 + 1$. By (4.8) and (4.10), we have

$$\|e^{-\gamma s}(F, \mathbf{G}, \mathbf{H})\|_{L_p(\mathbb{R}, W_q^{1,0}(\Omega))} + \|e^{-\gamma s}\nabla\mathbf{K}\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma s}\partial_s\mathbf{K}\|_{L_p(\mathbb{R}, \mathbf{W}_q^{-1}(\Omega))} \le C\mathcal{D}_t \tag{4.19}$$

with some constant C independent of t. By (4.18) and (4.19), we see that

$$(\theta, \mathbf{w}, \omega)(\cdot, s) = 0 \quad \text{for } s < 0. \tag{4.20}$$

In fact, we observe that

$$\|(\theta, \mathbf{w}, \omega)\|_{L_p((-\infty,0),W_q^{1,0}(\Omega))} \le \|e^{-\gamma s}(\theta, \mathbf{w}, \omega)\|_{L_p(\mathbb{R},W_q^{1,0}(\Omega))} \le \gamma^{-1}\mathcal{D}_t$$

for any $\gamma \ge \lambda_0 + 1$, so that we have (4.20) as $\gamma \to \infty$. Combining (4.18)–(4.20), we have

$$[\![(\theta, \mathbf{w}, \omega)]\!]_t \le Ce^{\gamma t}\mathcal{D}_t \tag{4.21}$$

for any $\gamma \ge \lambda_0 + 1$ with some constant C independent of γ. Moreover, since $[E_t f](\cdot s) = f(\cdot, s)$ for $s \in (0, t)$, by (4.11) and (4.20), the $(\theta, \mathbf{w}, \omega)$ is a solution to the equations:

$$\begin{cases} \partial_s\theta + \gamma_1 \mathrm{div}\,\mathbf{w} = f' & \text{in } \Omega \times (0, t) \\ \gamma_2\partial_s\mathbf{w} - \mathrm{Div}\,\mathbf{T}(\mathbf{w}, \gamma_3\theta) = \delta_1 \mathrm{Div}\,\omega + \mathbf{g}' & \text{in } \Omega \times (0, t) \\ \partial_s\omega + \delta_2\omega - g_\alpha(\nabla\mathbf{w}, \tau_1) = \delta_3\mathbf{D}(\mathbf{w}) + \mathbf{h}' & \text{in } \Omega \times (0, t) \\ (\mathbf{T}(\mathbf{w}, \gamma_3\theta) + \delta_1\omega)\mathbf{n} = \mathbf{k}' & \text{on } \Gamma_1 \times (0, t), \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \times (0, t), \\ (\theta, \mathbf{w}, \omega)|_{s=0} = (0, 0, 0) & \text{in } \Omega. \end{cases} \tag{4.22}$$

For $0 < t_1 < t_2 \le T$, let θ^{t_i}, $\mathbf{w}^{t_i}$, and ω^{t_i} be solutions of Eqs. (4.22) with $t = t_i$. By the uniqueness of solutions which follows from the solvability of the dual problem (cf. [25]), we have $(\theta^{t_1}, \mathbf{w}^{t_1}, \omega^{t_1}) = (\theta^{t_2}, \mathbf{w}^{t_2}, \omega^{t_2})$ for $s \in (0, t_1)$, so that if we set $(\theta, \mathbf{w}, \omega) = (\theta^T, \mathbf{w}^T, \omega^T)$, then we have $(\theta, \mathbf{w}, \omega) = (\theta^t, \mathbf{w}^t, \omega^t)$ for any $t \in (0, T]$. This completes the proof of Theorem 4.1.

5. A proof of the local wellposedness

In this section, we prove Theorem 1.2 by using the Banach fixed point theorem. In the sequel, we assume that $2 < p < \infty$, $N < q < \infty$, and that Ω is a uniform $W_q^{2-1/q}$ domain in $\mathbb{R}^N$ ($N \ge 2$). Let T and L be any positive numbers and let $\mathcal{I}_{L,T}$ be the space defined by

$$\begin{aligned} \mathcal{I}_{L,T} = \{(\theta, \mathbf{v}, \tau) \mid \theta \in W_p^1((0,T), W_q^1(\Omega)), \quad \mathbf{v} \in W_p^1((0,T), L_q(\Omega)) \cap L_p((0,T), W_q^2(\Omega)), \\ \tau \in W_p^1((0,T), W_q^1(\Omega)),\ (\theta, \mathbf{v}, \tau)|_{t=0} = (0, \mathbf{u}_0, 0) \text{ in } \Omega,\ [\![(\theta, \mathbf{v}, \tau)]\!]_T \le L\}. \end{aligned} \tag{5.1}$$

Since we choose $T > 0$ small enough and $L > 0$ large enough eventually, we may assume that $0 < T \le 1$ and $L \ge 1$. Moreover, we choose ρ_1 in (1.11) in such a way that $\|\tau_0\|_{W_q^1(\Omega)} \le R \le \rho_1$. Given $(\kappa, \mathbf{w}, \varphi) \in \mathcal{I}_{L,T}$, let θ, $\mathbf{v}$ and ψ be solutions to problem:

$$\begin{cases} \theta_t + (\rho_* + \theta_0)\mathrm{div}\,\mathbf{v} = F(\kappa, \mathbf{w}) & \text{in } \Omega \times (0, T), \\ (\rho_* + \theta_0)\mathbf{v}_t - \mathrm{Div}\,S(\mathbf{v}) + \nabla(P'(\rho_* + \theta_0)\theta) = \beta\,\mathrm{Div}\,\psi + \mathbf{g} + \mathbf{G}(\mathbf{w}, \kappa, \varphi) & \text{in } \Omega \times (0, T), \\ \psi_t + \gamma\psi - g_\alpha(\nabla\mathbf{u}, \tau_0) - \delta\mathbf{D}(\mathbf{v}) = -\gamma\tau_0 + \mathbf{L}(\mathbf{w}, \varphi) & \text{in } \Omega \times (0, T), \\ (\mathbf{S}(\mathbf{v}) - P'(\rho_* + \theta_0)\theta\mathbf{I} + \beta\psi)\mathbf{n} = \mathbf{h} + \mathbf{H}(\mathbf{w}, \kappa, \varphi) & \text{on } \Gamma_1 \times (0, T), \\ \mathbf{v} = 0 & \text{on } \Gamma_0 \times (0, T), \\ (\theta, \mathbf{v}, \tau)|_{t=0} = (0, \mathbf{u}_0, 0) & \text{in } \Omega. \end{cases} \tag{5.2}$$

In the sequel, C denotes generic constants independent of R and L, and C_R denotes generic constants independent of L. M_i denotes some special constants. The values of C and C_R may change from line to line. First, we estimate the right-hand side of (1.8). By the Sobolev inequality (cf. Lemma 2.2(1)), Hölder inequality and the identities: $\kappa(\cdot, t) = \int_0^t \partial_s\kappa(\cdot, s)\,ds$ and $\varphi(\cdot, t) = \int_0^t \partial_s\varphi(\cdot, s)\,ds$, we have

$$\begin{aligned} &\sup_{t\in(0,T)} \left\|\int_0^t \nabla\mathbf{w}(\cdot, s)ds\right\|_{L_\infty(\Omega)} \le M_1 T^{1/p'} L, \quad \sup_{t\in(0,T)} \left\|\int_0^t \nabla\mathbf{w}(\cdot, s)ds\right\|_{W_q^1(\Omega)} \le M_1 T^{1/p'} L \\ &\sup_{t\in(0,T)} \|\kappa(\cdot, s)\|_{L_\infty(\Omega)} \le M_1 T^{1/p'} L, \quad \sup_{t\in(0,T)} \|\kappa(\cdot, s)\|_{W_q^1(\Omega)} \le M_1 T^{1/p'} L \\ &\sup_{t\in(0,T)} \|\varphi(\cdot, s)\|_{L_\infty(\Omega)} \le M_1 T^{1/p'} L, \quad \sup_{t\in(0,T)} \|\varphi(\cdot, s)\|_{W_q^1(\Omega)} \le M_1 T^{1/p'} L \end{aligned} \tag{5.3}$$

with $p' = p/(p-1)$. To determine functions with respect to κ, φ and $\int_0^t \nabla \mathbf{w}(\cdot, s)\,ds$, in view of the range condition: $\frac{\rho_*}{2} < \rho_* + \theta_0 < 2\rho_*$ in Theorem 1.2 and (1.6), we choose T small enough in such a way that $M_1 T^{1/p'} L \le \rho_*/2$, $M_1 T^{1/p'} L \le \sigma$ and $M_1 T^{1/p'} L \le 1$, and then we have

$$\frac{\rho_*}{4} < \rho_* + \theta_0 + \ell\kappa < 4\rho_* \quad (\ell \in [0,1]), \qquad \sup_{t\in(0,T)} \left\| \int_0^t \nabla \mathbf{w}(\cdot, s)\,ds \right\|_{L_\infty(\Omega)} < \sigma. \tag{5.4}$$

Recall that $\|\theta_0\|_{W_q^1(\Omega)} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-\frac{1}{p})}(\Omega)} + \|\tau_0\|_{W_q^1(\Omega)} \le R$ (cf. Theorem 1.2 (1.14)). By (5.3) and (5.4) we have

$$\sup_{t\in(0,T)} \left\| V_i\left(\int_0^t \nabla \mathbf{w}(\cdot, s)\,ds\right)\right\|_{L_\infty(\Omega)} \le C T^{1/p'} L, \qquad \sup_{t\in(0,T)} \left\| \nabla \mathbf{W}\left(\int_0^t \nabla \mathbf{w}(\cdot, s)ds\right)\right\|_{W_q^1(\Omega)} \le C T^{1/p'} L$$
$$\sup_{t\in(0,T)} \left\| \nabla \int_0^1 P''(\rho_* + \theta_0 + \ell\kappa)(1-\ell)d\ell \right\|_{L_q(\Omega)} \le C(R + T^{1/p'} L) \tag{5.5}$$

where $i = D$, div, W, and $\mathbf{W} = \mathbf{W}(\mathbf{K})$ is any matrix of functions with respect to $\mathbf{K}$. By Lemma 2.2(1), (2.10), (5.3), (5.4) and (5.5), we have

$$\begin{aligned} &\|(0, \mathbf{g}, -\gamma\tau_0)\|_{L_p((0,T),W_q^{1,0}(\Omega))} + \|\mathbf{h}\|_{L_p((0,T),W_q^1(\Omega))} + \|\partial_t \mathbf{h}\|_{L_p((0,T),\mathbf{W}_q^{-1}(\Omega))} \le CRT^{1/p}, \\ &\|(F(\kappa, \mathbf{w}), \mathbf{G}(\mathbf{w}, \kappa, \varphi), \mathbf{L}(\mathbf{w}, \varphi))\|_{L_p((0,T),W_q^{1,0}(\Omega))} \le C(L+R)^2(T^{1/p'} + T^{1/p}), \\ &\|\mathbf{H}(\mathbf{w}, \kappa, \varphi)\|_{L_p((0,T),W_q^1(\Omega))} \le C(L+R)^2(T^{1/p'} + T^{1/p}), \end{aligned} \tag{5.6}$$

where we have used the fact that $\partial_t \mathbf{h} = 0$.

To obtain the following estimates,

$$\sup_{t\in(0,T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le C(\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W_q^2(\Omega))} + \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}) \tag{5.7}$$

we use the embedding relation:

$$L_p((0,\infty), X_1) \cap W_p^1((0,\infty), X_0) \subset BUC((0,\infty), [X_0, X_1]_{1-1/p,p}) \tag{5.8}$$

for any two Banach spaces X_0 and X_1 such that X_1 dense in X_0 and $1 < p < \infty$ (cf. [1]). In fact, as was seen in Section 4, let $\tilde{\mathbf{u}}_0 \in B_{q,p}^{2(1-1/p)}(\Omega_1)$ be an extension of $\mathbf{u}_0$ such that $\tilde{\mathbf{u}}_0 = \mathbf{u}_0$ on Ω and $\|\tilde{\mathbf{u}}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega_1)} \le C\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$, and then there exists a $\mathbf{Z} \in W_p^1((0,\infty), L_q(\Omega)^N) \cap L_p((0,\infty), W_q^2(\Omega)^N)$ which satisfies the equations:

$$\partial_t \mathbf{Z} + \lambda_0 \mathbf{Z} - \mu\Delta\mathbf{Z} = 0 \quad \text{in } \Omega_1 \times (0,\infty), \qquad \mathbf{Z}|_{\Gamma_0} = 0, \qquad \mathbf{Z}|_{t=0} = \tilde{\mathbf{u}}_0 \quad \text{in } \Omega_1,$$

and possesses the estimate:

$$\|\partial_t \mathbf{Z}\|_{L_p((0,\infty),L_q(\Omega_1))} + \|\mathbf{Z}\|_{L_p((0,\infty),W_q^2(\Omega_1))} \le C\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \tag{5.9}$$

with some constant C. Let $\mathbf{z} = \mathbf{w} - \mathbf{Z}$. Since $\mathbf{z}|_{t=0} = 0$, by (4.8) and (4.9) we have

$$\begin{aligned} &\|E_T \mathbf{z}\|_{W_p^1((0,\infty),L_q(\Omega))} + \|E_T \mathbf{z}\|_{L_p((0,\infty),W_q^2(\Omega))} \le C\{\|\mathbf{z}\|_{W_p^1((0,T),L_q(\Omega))} + \|\mathbf{z}\|_{L_p((0,T),W_q^2(\Omega))}\} \\ &\quad \le C\{\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W_q^2(\Omega))} + \|\partial_t \mathbf{Z}\|_{L_p((0,\infty),L_q(\Omega))} + \|\mathbf{Z}\|_{L_p((0,\infty),W_q^2(\Omega))}\}. \end{aligned} \tag{5.10}$$

Thus, noting that $\mathbf{w} = \mathbf{Z} + E_T \mathbf{z}$ for $t \in (0,T)$ and using (5.8), we have

$$\begin{aligned} &\sup_{t\in(0,T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \le \sup_{t\in(0,\infty)} \|\mathbf{Z}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \sup_{t\in(0,\infty)} \|E_T \mathbf{z}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &\quad \le C\{\|\partial_t \mathbf{w}\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W_q^2(\Omega))} + \|\partial_t \mathbf{Z}\|_{L_p(\mathbb{R}_+,L_q(\Omega))} + \|\mathbf{Z}\|_{L_p(\mathbb{R}_+,W_q^2(\Omega))}\}, \end{aligned} \tag{5.11}$$

which, combined with (5.9), furnishes (5.7). Since $B^{2(1-1/p)}_{q,p}(\Omega) \subset W^1_q(\Omega)$ as follows from the assumption: $2 < p < \infty$ by (5.7) and the fact: $\sup_{t\in(0,T)} \| \int_0^t \nabla \mathbf{w}(\cdot, s) ds\|_{L_\infty(\Omega)} \le 1$, we have

$$\begin{aligned}\sup_{t\in(0,T)} \|\mathbf{w}(\cdot,t)\|_{W^1_q(\Omega)} &\le \sup_{t\in(0,T)} \|\mathbf{w}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \\ &\le C\{\|\mathbf{w}_t\|_{L_p((0,T),L_q(\Omega))} + \|\mathbf{w}\|_{L_p((0,T),W^2_q(\Omega))} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\}.\end{aligned}$$

By (5.1) we have

$$\sup_{t\in(0,T)} \|\mathbf{w}(\cdot,t)\|_{W^1_q(\Omega)} \le C(L+R). \tag{5.12}$$

Writing $V'_j(\mathbf{K}) = \partial V_j/\partial \mathbf{K}$ for $j = D$ and $=$ div , we have

$$\begin{aligned}\partial_t \mathbf{H}(\mathbf{w},\kappa,\varphi) = &-\left\{\mu V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\partial_t\mathbf{w} + \mu\left(V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w}\right. \\ &\left.+(\nu-\mu)\left(V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\partial_t\nabla\mathbf{w} + \left(V'_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w}\right)\mathbf{I}\right\}\mathbf{n} \\ &-\left\{\mu(\mathbf{D}(\partial_t\mathbf{w}) + V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\partial_t\mathbf{w} + \left(V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w})\right\}V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n} \\ &-\left\{\mu\left\{\left(\mathbf{D}(\mathbf{w}) + V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\right\}V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right\}\mathbf{n} \\ &-(\nu-\mu)\left\{(\mathrm{div}\,(\partial_t\mathbf{w}) + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\partial_t\mathbf{w} + \left(V'_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\nabla\mathbf{w})\mathbf{I}\right\}V_D\left(\int_0^t \nabla\mathbf{v}\,ds\right)\mathbf{n} \\ &-\left\{(\nu-\mu)\left\{\left(\mathrm{div}\,\mathbf{w} + V_{\mathrm{div}}\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right)\mathbf{I}\right\}V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right\}\mathbf{n} \\ &+\left(2\int_0^1 P''(\rho_*+\theta_0+\ell\kappa)(1-\ell)\,d\ell\,\kappa\partial_t\kappa + \int_0^1 P'''(\rho_*+\theta_0+\ell\kappa)(1-\ell)\ell\,d\ell\,\kappa^2\partial_t\kappa\right)\mathbf{n} \\ &+\left\{(P(\rho_*+\theta_0+\kappa)-P(\rho_*))V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w} + P'(\rho_*+\theta_0+\kappa)\partial_t\kappa V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\right\}\mathbf{n} \\ &-\left\{\beta\partial_t\varphi V_D\left(\int_0^t \nabla\mathbf{w}\,ds\right) + \beta(\varphi+\tau_0)V'_D\left(\int_0^t \nabla\mathbf{w}\,ds\right)\nabla\mathbf{w}\right\}\mathbf{n}.\end{aligned}$$

Applying Lemma 2.2 and using (5.12), (5.3) and (5.4), we have

$$\|\partial_t \mathbf{H}(\mathbf{w},\kappa,\varphi)\|_{L_p((0,T),\mathbf{W}^{-1}_q(\Omega))} \le C(L+R)^2(T^{1/p'} + T^{1/p}). \tag{5.13}$$

Thus, applying Theorem 4.1 to problem (5.2) and using (5.6) and (5.13), we have

$$[\![(\theta,\mathbf{v},\psi)]\!]_T \le C_R(L+R)^2(T^{1/p'} + T^{1/p}). \tag{5.14}$$

Choosing $T > 0$ so small that $C_R(L+R)^2(T^{1/p'} + T^{1/p}) \le L$ in (5.14), we have

$$[\![(\theta,\mathbf{v},\psi)]\!]_T \le L. \tag{5.15}$$

Let us define a map Φ by $\Phi(\kappa,\mathbf{w},\varphi) = (\theta,\mathbf{v},\psi)$, and then by (5.15) Φ is a map from $\mathcal{I}_{L,T}$ into itself. For $(\kappa_i,\mathbf{w}_i,\varphi_i) \in \mathcal{I}_{L,T}$ $(i = 1,2)$ let $(\theta,\mathbf{v},\psi) = \Phi(\kappa_1,\mathbf{w}_1,\varphi_1) - \Phi(\kappa_2,\mathbf{w}_2,\varphi_2)$, and let

$$\begin{aligned}\mathcal{F} &= F(\kappa_1,\mathbf{w}_1) - F(\kappa_2,\mathbf{w}_2), \quad \mathcal{G} = \mathbf{G}(\mathbf{w}_1,\kappa_1,\varphi_1) - \mathbf{G}(\mathbf{w}_2,\kappa_2,\varphi_2), \\ \mathcal{L} &= \mathbf{L}(\mathbf{w}_1,\varphi_1) - \mathbf{L}(\mathbf{w}_2,\varphi_2), \quad \mathcal{H} = \mathbf{H}(\mathbf{w}_1,\kappa_1,\varphi_1) - \mathbf{H}(\mathbf{w}_2,\kappa_2,\varphi_2),\end{aligned}$$

then by (5.2) we have

$$\begin{cases} \theta_t + (\rho_* + \theta_0)\operatorname{div} \mathbf{v} = \mathcal{F} & \text{in } \Omega \times (0, T), \\ (\rho_* + \theta_0)\mathbf{v}_t - \operatorname{Div} S(\mathbf{v}) + \nabla(P'(\rho_* + \theta_0)\theta) = \beta \operatorname{Div} \psi + \mathcal{G} & \text{in } \Omega \times (0, T), \\ \psi_t + \gamma\psi - g_\alpha(\nabla \mathbf{u}, \tau_0) - \delta \mathbf{D}(\mathbf{v}) = \mathcal{L} & \text{in } \Omega \times (0, T), \\ (\mathbf{S}(\mathbf{v}) - P'(\rho_* + \theta_0)\theta\mathbf{I} + \beta\psi)\mathbf{n} = \mathcal{H} & \text{on } \Gamma_1 \times (0, T), \\ \mathbf{v} = 0 & \text{on } \Gamma_0 \times (0, T), \\ (\theta, \mathbf{v}, \tau)|_{t=0} = (0, 0, 0) & \text{in } \Omega. \end{cases} \tag{5.16}$$

Since

$$\sup_{0 \in (0,T)} \|(\mathbf{v}_1 - \mathbf{v}_2)(\cdot, t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq C(\|\partial_t(\mathbf{v}_1 - \mathbf{v}_2)\|_{L_p((0,T), L_q(\Omega))} + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L_p((0,T), W^2_q(\Omega))})$$

as follows from (5.7), employing the same argumentation as in proving (5.6) and (5.13) we have

$$\begin{aligned} &\|(\mathcal{F}, \mathcal{G}, \mathcal{L})\|_{L_p((0,T), W^{1,0}_q(\Omega))} + \|\mathcal{K}\|_{L_p((0,T), W^1_q(\Omega))} + \|\partial_t \mathcal{K}\|_{L_p((0,T), \mathbf{W}^{-1}_q(\Omega))} \\ &\quad \leq C(R + L)(T^{1/p'} + T^{1/p})[\![(\kappa_1, \mathbf{w}_1, \varphi_1) - (\kappa_2, \mathbf{w}_2, \varphi_2)]\!]_T. \end{aligned}$$

Thus, applying Theorem 4.1 to Eqs. (5.16), we have

$$[\![\Phi(\kappa_1, \mathbf{w}_1, \varphi_1) - \Phi(\kappa_2, \mathbf{w}_2, \varphi_2)]\!]_T \leq C_R(R + L)(T^{1/p'} + T^{1/p})[\![(\kappa_1, \mathbf{w}_1, \varphi_1) - (\kappa_2, \mathbf{w}_2, \varphi_2)]\!]_T$$

with some constant C_R depending on R. Choosing $T > 0$ so small that $C_R(R + L)(T^{1/p'} + T^{1/p}) \leq 1/2$, we see that Φ is a contraction map on $\mathcal{I}_{L,T}$, and therefore by the Banach fixed point theorem we have a unique fixed point $(\theta, \mathbf{v}, \pi)$ of the map Φ, which solves Eqs. (1.8) uniquely. This completes the proof of Theorem 1.2.

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