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ON THE BICHROMATIC WAVE PROPAGATION OVER VARYING BOTTOM

Mashuri and Rina Reorita

Department of Mathematics
Universitas Jenderal Soedirman
Jl. Dr. Soeparno No. 61 Purwokerto, 53122
Indonesia
e-mail: mashuri_unsoed@yahoo.com

Abstract

In this paper, we discuss the propagation of water wave over varying bottom using AB equation as governing equation. The AB equation is an improvement of the KdV equation and can be interpreted as higher KdV equation for wave above finite depth and under certain approximation it becomes the KdV equation [3]. We solve the equation using asymptotic method with bichromatic wave as a signal input, then we apply the varying bottom to the operators of AB equation that contain exact dispersion relation. We observe the propagation of the wave that produced over a slope bottom and find the position of maximal amplitude and amplification of the wave. Comparison the result with the AB equation for flat bottom to study the different characteristic of each other will be studied.

1. Introduction

The wave propagation is an interesting problem that we discuss today,

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especially in determining the position of maximal amplitude of the wave and its amplification. By using asymptotic method, the position maximal amplitude of the wave and its amplification was discussed for AB equation over flat bottom by Mashuri et al. [5]. The AB equation that used in the literature is improvement of the KdV type equation and can be interpreted as a higher KdV type for wave above finite depth and under certain approximation it becomes the KdV equation [2, 3] and [5]. In this paper, we use the AB equation to study bichromatic wave propagation but for varying bottom and compare the results with the constant bottom, especially in determining the position of maximal amplitude of the wave and its amplification using maximal temporal amplitude (MTA). The paper is organized as follows. In Section 2, we will discuss mathematical model equation that used in the paper. Here, the AB equation that proposed by Groesen and Andonowati [3] is modified for varying bottom. The dispersion relation of the AB equation is replaced for varying bottom. It means that the dispersion relation is depended on space. The review of the solution of AB equation using asymptotic method is discussed in Section 3. In Section 4, we discuss the propagation of the wave based on the asymptotic solution. In the section also we discuss the maximal temporal amplitude and the amplitude amplification factor of the wave. Conclusion will be given in Section 5.

2. Mathematical Model Equation

In the simplest second order wave equation, the elevation of surface water wave on the constant depth h_0 is given by $\eta(x, t)$ with the propagation of wave is described by the equation $\partial_t^2 \eta = c_0^2 \partial_x^2 \eta$, where $c_0 = \sqrt{gh_0}$. The unidirectional of the equation is given by $\partial_t \eta = -c_0 \partial_x \eta$. Above varying bottom $h(x)$, the wave equation becomes $\partial_t^2 \eta = c(x)^2 \partial_x^2 \eta$, where $c(x) = \sqrt{gh(x)}$, while in the linear water wave equation with the exact dispersion relation $\omega = \Omega(k, h_0)$ for the flat bottom h_0, the unidirectional wave is described by $\partial_t \eta = -\Omega(k, h_0)\eta$. Above varying bottom $h(x)$, the exact dispersion relation becomes $\omega = \Omega(k, h(x))$.

In this section, we use AB-equation that proposed by Groesen and Andonowati [3]. The equation describes the unidirectional water wave on the flat bottom h_0 and is given by

$$\partial_t \eta = -\sqrt{g} A\left[\eta + \frac{1}{2} A(\eta A \eta) - \frac{1}{4}(A\eta)^2 + \frac{1}{2} B(\eta B \eta) + \frac{1}{4}(B\eta)^2\right] \quad (1)$$

with η represents elevation of wave, $A = \frac{\partial_x C}{\sqrt{g}}$ and $B = \sqrt{g} C^{-1}$ represent pseudo-differential operator with the symbol $\hat{C}(k) = \frac{\Omega(k)}{k}$, $\Omega(k) = c_0 k \sqrt{\frac{\tanh(kh_0)}{kh_0}}$, $c_0 = \sqrt{gh_0}$ and g represents acceleration of gravitation.

In the paper of [3] and [5], the exact dispersion relation of the AB equation (1) is given by $\omega = \Omega(k, h_0)$. The dispersion relation is for the wave over the flat bottom. For the varying bottom $h(x)$, it becomes

$$\Omega(k, h(x)) = \sqrt{gh(x)}\, k \sqrt{\frac{\tanh(kh(x))}{kh(x)}} = \sqrt{gk \tanh(kh(x))}. \quad (2)$$

3. Asymptotic Solution of the Mathematical Model

The third order asymptotic solution of the wave equation for flat bottom case was discussed in [4] where the asymptotic solution founded for the mKdV equation and in [5] where the solution also obtained for AB equation. The solution is given by

$$\eta = \eta^{(1)} + \eta^{(2)} - \eta^{(2)}_{fw} + \eta^{(3)} - \eta^{(3)}_{fw} \quad (3)$$

with $\eta^{(i)}$ is the solution of order i. $\eta^{(i)}_{fw}$ is free wave of order i. The free waves are taken here if we want to prescribe the bichromatic wave as a signal input, it means that the bound waves should be compensated by 2nd- and 3rd- order free waves. As in [5], the bichromatic wave is chosen for the first order of the solution $\eta^{(1)}$ and written by

$$\eta^{(1)} = ae^{i\theta_+} + ae^{i\theta_-} + c.c \tag{4}$$

with $\theta_\pm = k_\pm x - \omega_\pm t$ and $c.c$ means conjugate complex. The dispersion relation of the solution is given by $\omega_\pm = \Omega(k_\pm, h(x))$, while the second and third order solutions are, respectively,

$$\eta^{(2)} = a_{21}e^{2i\theta_+} + a_{22}e^{2i\theta_-} + a_{23}e^{i(\theta_+ + \theta_-)} + a_{24}e^{i(\theta_+ - \theta_-)} + c.c \tag{5}$$

and

$$\eta^{(3)} = a_{31}e^{3i\theta_+} + a_{32}e^{3i\theta_-} + a_{33}e^{i(2\theta_+ + \theta_-)} + a_{34}e^{i(2\theta_- + \theta_+)}$$
$$+ a_{35}e^{i(2\theta_+ - \theta_-)} + a_{36}e^{i(2\theta_- - \theta_+)} + c.c \tag{6}$$

with their coefficients for constant depth h_0 are given in Mashuri et al. [5]. For the varying bottom, the constant depth is replaced by the varying depth $h(x)$, and the second order coefficient of the solution will become

$$a_{21} = \frac{\alpha_1}{\sqrt{g}A(2k_+^{(0)}) - 2i\omega_+}, \qquad a_{22} = \frac{\alpha_2}{\sqrt{g}A(2k_-^{(0)}) - 2i\omega_-}$$

$$a_{23} = \frac{\alpha_3}{\sqrt{g}A(k_+^{(0)} + k_-^{(0)}) - i(\omega_+ + \omega_-)},$$

$$a_{24} = \frac{\alpha_4}{\sqrt{g}A(k_+^{(0)} - k_-^{(0)}) - i(\omega_+ - \omega_-)}$$

with

$$\alpha_1 = -\sqrt{g}a^2\Big[-\frac{1}{4}A^2(k_+^{(0)})A(2k_+^{(0)}) + \frac{1}{2}A(k_+^{(0)})A^2(2k_+^{(0)})$$
$$+ \frac{1}{4}B^2(k_+^{(0)})A(2k_+^{(0)}) + \frac{1}{2}B(k_+^{(0)})B(2k_+^{(0)})A(2k_+^{(0)})\Big],$$

$$\alpha_2 = -\sqrt{g}a^2\Big[-\frac{1}{4}A^2(k_-^{(0)})A(2k_-^{(0)}) + \frac{1}{2}A(k_-^{(0)})A^2(2k_-^{(0)})$$
$$+ \frac{1}{4}B^2(k_-^{(0)})A(2k_-^{(0)}) + \frac{1}{2}B(k_-^{(0)})B(2k_-^{(0)})A(2k_-^{(0)})\Big],$$

$$\alpha_3 = -\sqrt{g}a^2\Big[-\frac{1}{2}A(k_+^{(0)})A(k_-^{(0)})A(k_+^{(0)} + k_-^{(0)}) + \frac{1}{2}(A(k_+^{(0)})$$

$$+ A(k_-^{(0)}))A^2(k_+^{(0)} + k_-^{(0)}) + \frac{1}{2}B(k_+^{(0)})B(k_-^{(0)})A(k_+^{(0)} + k_-^{(0)})$$

$$+ \frac{1}{2}(B(k_+^{(0)}) + B(k_-^{(0)}))B(k_+^{(0)} + k_-^{(0)})A(k_+^{(0)} + k_-^{(0)})\Big],$$

$$\alpha_4 = -\sqrt{g}a^2\Big[\frac{1}{2}A(k_+^{(0)})A(k_-^{(0)})A(k_+^{(0)} - k_-^{(0)}) + \frac{1}{2}(A(k_+^{(0)})$$

$$- A(k_-^{(0)}))A^2(k_+^{(0)} - k_-^{(0)}) + \frac{1}{2}B(k_+^{(0)})B(k_-^{(0)})A(k_+^{(0)} - k_-^{(0)})$$

$$+ \frac{1}{2}(B(k_+^{(0)}) + B(k_-^{(0)}))B(k_+^{(0)} - k_-^{(0)})A(k_+^{(0)} - k_-^{(0)})\Big],$$

while for the third order solution, the coefficients of the solution will become

$$a_{31} = \frac{\beta_1}{-3i\omega_+ + \sqrt{g}A(3k_+^{(0)})}, \qquad a_{32} = \frac{\beta_2}{-3i\omega_- + \sqrt{g}A(3k_-^{(0)})},$$

$$a_{33} = \frac{\beta_3}{-i(2\omega_+ + \omega_-) + \sqrt{g}A(2k_+^{(0)} + k_-^{(0)})},$$

$$a_{34} = \frac{\beta_4}{-i(2\omega_- + \omega_+) + \sqrt{g}A(2k_-^{(0)} + k_+^{(0)})},$$

$$a_{35} = \frac{\beta_5}{-i(2\omega_+ - \omega_-) + \sqrt{g}A(2k_+^{(0)} - k_-^{(0)})},$$

$$a_{36} = \frac{\beta_6}{-i(2\omega_- - \omega_+) + \sqrt{g}A(2k_-^{(0)} - k_+^{(0)})}$$

with

$$\beta_1 = -\sqrt{g}\Big[-\frac{1}{2}aa_{21}A(k_+^{(0)})A(2k_+^{(0)}) + \frac{1}{2}aa_{21}(A(k_+^{(0)}) + A(2k_+^{(0)}))A(3k_+^{(0)})$$

$$+ \frac{1}{2}aa_{21}B(k_+^{(0)})B(2k_+^{(0)}) + \frac{1}{2}aa_{21}(B(k_+^{(0)}) + B(2k_+^{(0)}))B(3k_+^{(0)})\Big]$$

$$\cdot A(3k_+^{(0)})e^{3i\theta_+},$$

$$\beta_2 = -\sqrt{g}\Big[-\frac{1}{2}aa_{22}A(k_-^{(0)})A(2k_-^{(0)}) + \frac{1}{2}aa_{22}(A(k_-^{(0)}) + A(2k_-^{(0)}))A(3k_-^{(0)})$$

$$+ \frac{1}{2}aa_{22}B(k_-^{(0)})B(2k_-^{(0)}) + \frac{1}{2}aa_{22}(B(k_-^{(0)}) + B(2k_-^{(0)}))B(3k_-^{(0)})\Big]$$

$$\cdot A(3k_-^{(0)})e^{3i\theta_-},$$

$$\beta_3 = -\sqrt{g}\Big[-\frac{1}{2}aa_{23}A(k_+^{(0)})A(k_+^{(0)} + k_-^{(0)}) - \frac{1}{2}aa_{21}A(k_-^{(0)})A(2k_+^{(0)})$$

$$+ \left(\frac{1}{2}aa_{23}(A(k_+^{(0)}) + A(k_+^{(0)} + k_-^{(0)})) + \frac{1}{2}aa_{21}(A(k_-^{(0)}) + A(2k_+^{(0)}))\right)$$

$$\cdot A(2k_+^{(0)} + k_-^{(0)}) + \frac{1}{2}aa_{23}B(k_+^{(0)})B(k_+^{(0)} + k_-^{(0)})$$

$$- \frac{1}{2}aa_{21}B(k_-^{(0)})B(2k_+^{(0)}) + \Big(\frac{1}{2}aa_{23}(B(k_+^{(0)}) + B(k_+^{(0)} + k_-^{(0)}))$$

$$+ \frac{1}{2}aa_{21}(B(k_-^{(0)}) + B(2k_+^{(0)}))\Big)B(2k_+^{(0)} + k_-^{(0)})\Big]$$

$$\cdot A(2k_+^{(0)} + k_-^{(0)})e^{i(2\theta_+ + \theta_-)},$$

$$\beta_4 = -\sqrt{g}\Big[-\frac{1}{2}aa_{23}A(k_-^{(0)})A(k_+^{(0)} + k_-^{(0)}) - \frac{1}{2}aa_{22}A(k_+^{(0)})A(2k_-^{(0)})$$

$$+ \left(\frac{1}{2}aa_{23}(A(k_-^{(0)}) + A(k_+^{(0)} + k_-^{(0)})) + \frac{1}{2}aa_{22}(A(k_+^{(0)}) + A(2k_-^{(0)}))\right)$$

$$\cdot A(2k_-^{(0)} + k_+^{(0)}) + \frac{1}{2}aa_{23}B(k_-^{(0)})B(k_+^{(0)} + k_-^{(0)})$$

$$- \frac{1}{2}aa_{22}B(k_+^{(0)})B(2k_-^{(0)}) + \Big(\frac{1}{2}aa_{23}(B(k_-^{(0)}) + B(k_+^{(0)} + k_-^{(0)}))$$

$$+ \frac{1}{2}aa_{22}(B(k_+^{(0)}) + B(2k_-^{(0)}))\Big)B(2k_-^{(0)} + k_+^{(0)})\Big]$$

$$\cdot A(2k_-^{(0)} + k_+^{(0)})e^{i(2\theta_- + \theta_+)},$$

$$\beta_5 = -\sqrt{g}\Big[-\frac{1}{2}aa_{24}A(k_+^{(0)})A(k_+^{(0)} - k_-^{(0)}) + \frac{1}{2}aa_{21}A(k_-^{(0)})A(2k_+^{(0)})$$

$$+\left(\frac{1}{2}aa_{24}(A(k_+^{(0)}) + A(k_+^{(0)} - k_-^{(0)})) + \frac{1}{2}aa_{21}(A(2k_+^{(0)}) - A(k_-^{(0)}))\right)$$

$$\cdot A(2k_+^{(0)} - k_-^{(0)}) + \frac{1}{2}aa_{24}B(k_+^{(0)})B(k_+^{(0)} - k_-^{(0)})$$

$$+\frac{1}{2}aa_{21}B(k_-^{(0)})B(2k_+^{(0)}) + \left(\frac{1}{2}aa_{24}(B(k_+^{(0)}) + B(k_+^{(0)} - k_-^{(0)}))\right.$$

$$\left.+\frac{1}{2}aa_{21}(B(2k_+^{(0)}) - B(k_-^{(0)}))\right)B(2k_+^{(0)} - k_-^{(0)})\Big]$$

$$\cdot A(2k_+^{(0)} - k_-^{(0)})e^{i(2\theta_+ - \theta_-)},$$

$$\beta_6 = -\sqrt{g}\Big[\frac{1}{2}aa_{24}A(k_-^{(0)})A(k_+^{(0)} - k_-^{(0)}) + \frac{1}{2}aa_{22}A(k_+^{(0)})A(2k_-^{(0)})$$

$$+\left(\frac{1}{2}aa_{24}(A(k_-^{(0)}) - A(k_+^{(0)} - k_-^{(0)})) + \frac{1}{2}aa_{22}(A(2k_-^{(0)}) - A(k_+^{(0)}))\right)$$

$$\cdot A(2k_-^{(0)} - k_+^{(0)}) + \frac{1}{2}aa_{24}B(k_-^{(0)})B(k_+^{(0)} - k_-^{(0)})$$

$$+\frac{1}{2}aa_{22}B(k_+^{(0)})B(2k_-^{(0)}) + \left(\frac{1}{2}aa_{24}(B(k_-^{(0)}) + B(k_+^{(0)} + k_-^{(0)}))\right.$$

$$\left.+\frac{1}{2}aa_{22}(B(k_+^{(0)}) + B(2k_-^{(0)}))\right)B(2k_-^{(0)} - k_+^{(0)})\Big]$$

$$\cdot A(2k_-^{(0)} - k_+^{(0)})e^{i(2\theta_- + \theta_+)},$$

$$\beta_7 = -\sqrt{g}\Big[\frac{1}{2}aa_{21}A(k_+^{(0)})A(2k_+^{(0)}) - \frac{1}{2}aa_{24}A(k_-^{(0)})A(k_+^{(0)} - k_-^{(0)})$$

$$+\frac{1}{2}aa_{23}A(k_-^{(0)})A(k_+^{(0)} + k_-^{(0)}) + \left(\frac{1}{2}aa_{21}(A(2k_+^{(0)}) - A(k_+^{(0)}))\right.$$

$$+\frac{1}{2}aa_{24}(A(k_+^{(0)} - k_-^{(0)}) + A(k_-^{(0)}))$$

$$+\frac{1}{2}aa_{23}(A(k_+^{(0)}+k_-^{(0)})-A(k_-^{(0)}))\Big)A(k_+^{(0)})$$

$$+\frac{1}{2}aa_{21}B(k_+^{(0)})B(2k_+^{(0)})+\frac{1}{2}aa_{24}B(k_-^{(0)})B(k_+^{(0)}-k_-^{(0)})$$

$$+\frac{1}{2}aa_{23}B(k_-^{(0)})B(k_+^{(0)}+k_-^{(0)})+\Big(\frac{1}{2}aa_{21}(B(2k_+^{(0)})+B(k_+^{(0)}))$$

$$+\frac{1}{2}aa_{24}(B(k_+^{(0)}-k_-^{(0)})+B(k_-^{(0)}))$$

$$+\frac{1}{2}aa_{23}(B(k_+^{(0)}+k_-^{(0)})+B(k_-^{(0)}))\Big)B(k_+^{(0)})\Big]A(k_+^{(0)})e^{i\theta_+},$$

$$\beta_8=-\sqrt{g}\Big[\frac{1}{2}aa_{22}A(k_-^{(0)})A(2k_-^{(0)})+\frac{1}{2}aa_{24}A(k_+^{(0)})A(k_+^{(0)}-k_-^{(0)})$$

$$+\frac{1}{2}aa_{23}A(k_+^{(0)})A(k_+^{(0)}+k_-^{(0)})$$

$$+\Big(\frac{1}{2}aa_{22}(A(2k_-^{(0)})-A(k_-^{(0)}))+\frac{1}{2}aa_{24}(-A(k_+^{(0)}-k_-^{(0)})$$

$$+A(k_+^{(0)}))+\frac{1}{2}aa_{23}(A(k_+^{(0)}+k_-^{(0)})-A(k_+^{(0)}))\Big)A(k_+^{(0)})$$

$$+\frac{1}{2}aa_{22}B(k_-^{(0)})B(2k_-^{(0)})+\frac{1}{2}aa_{24}B(k_+^{(0)})B(k_+^{(0)}-k_-^{(0)})$$

$$+\frac{1}{2}aa_{23}B(k_+^{(0)})B(k_+^{(0)}+k_-^{(0)})+\Big(\frac{1}{2}aa_{22}(B(2k_-^{(0)})+B(k_-^{(0)}))$$

$$+\frac{1}{2}aa_{24}(B(k_+^{(0)}-k_-^{(0)})+B(k_+^{(0)}))$$

$$+\frac{1}{2}aa_{23}(B(k_+^{(0)}+k_-^{(0)})+B(k_+^{(0)}))\Big)B(k_+^{(0)})\Big]A(k_-^{(0)})e^{i\theta_-}.$$

4. Propagation of Bichromatic Wave over the Sloping Bottom

In this section, we discuss the propagation of the bichromatic wave over sloping bottom which is given in the following figure:

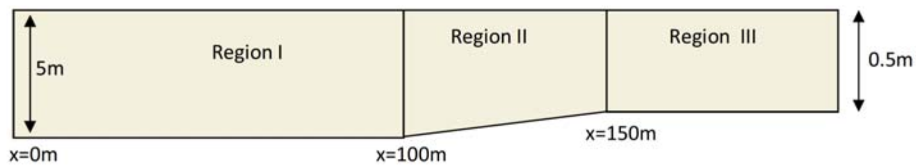

Figure 1. The sloping bottom condition of the wave tank with 5 m depth for region I (0-100 m) and 0.5 m for region III (150-200 m).

The propagation of the bichromatic wave in region I at the position $x = 90$ m for 5 m depth is given in Figure 2. The amplitude of the case is 0,193 m and higher than the signal input which has amplitude 0,16 m. For this case we take frequency $\overline{\omega} = \frac{\omega_+ + \omega_-}{2} = 3.1436\,\text{hz}$ and $\nu = \frac{\omega_+ - \omega_-}{2} = 1.575\,\text{hz}$.

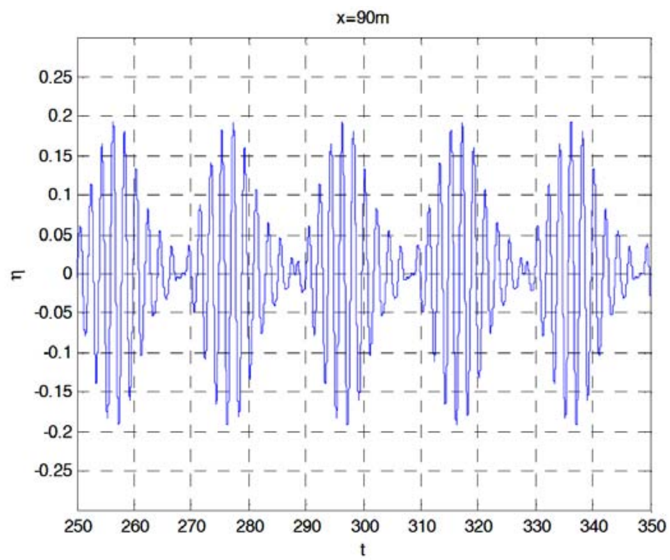

Figure 2. Signal at position 90 m from source.

The contribution of 2nd and 3rd order solutions is given in Figure 3.

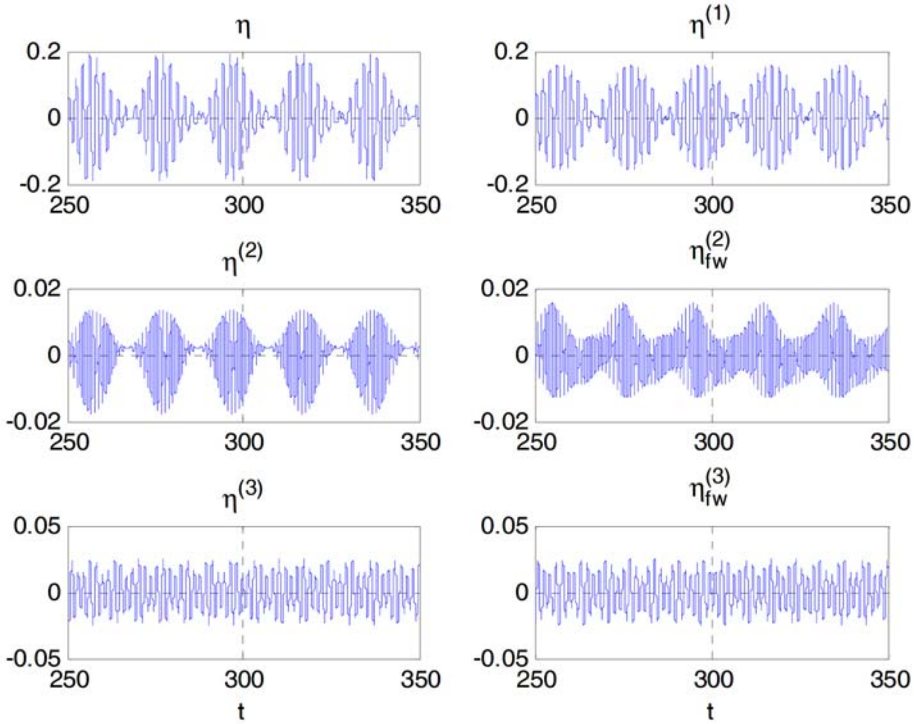

Figure 3. The contribution of the first, second, and third order solutions of the wave.

Comparison between bichromatic wave that obtained for the constant depth (5 m) and sloping bottom is given in Figure 4. The left side of the figure is for flat bottom and the right side for varying bottom at position 0 m, 70 m, 110 m, and 160 m source. The high of wave that obtained in the constant depth is lower than the varying depth with ratio is 1 : 10.

Figure 4. Signals at position 0 m, 70 m, 110 m, 160 m for constant depth (left) and varying depth (right).

5. Maximal Temporal Amplitude

Maximal temporal amplitude (MTA) is a maximal wave elevation in each position of the tank as long as propagation time. MTA is proposed by Andonowati and Groesen [1] when they determined the maximal amplitude position for optic waves. In [4] and [5], the authors also use the formula for determining the maximal amplitude position and its amplification for wave propagation over flat bottom. The maximal temporal amplitude is defined as follows:

$$M(x) = \max_{t \in [0, T]} \eta(x, t), \tag{7}$$

where $\eta(x, t)$ represents wave elevation and $[0, T]$ is time of propagation.

Maximal temporal amplitude (MTA) for the case that discussed in Section 4 with the constant depth 5 m is given in Figure 5, while for the varying bottom, the MTA is given in Figure 6.

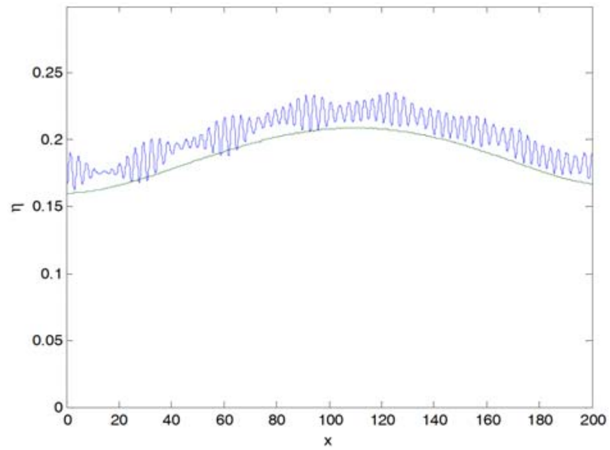

Figure 5. Maximal temporal amplitude for flat bottom (5 m depth) with (blue) and without (green) the second order solution.

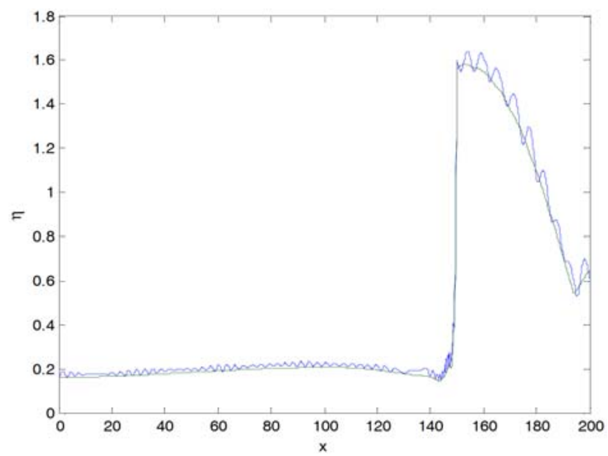

Figure 6. Maximal temporal amplitude for varying bottom with (blue) and without (green) the second order solution.

In the AB equation with flat bottom, the position of maximal amplitude of wave is happened at $x = 112\,\mathrm{m}$ with amplification 1,47, while for the varying bottom the maximal amplitude is happened at position $x = 154\,\mathrm{m}$, with amplification 10,3. The effect of depth water is very significant for the case especially for the position and high of the wave. The comparison between MTA of flat bottom and the varying bottom is given in Figure 7. From the figure we can see that the maximal amplitude is happened at the lower depth.

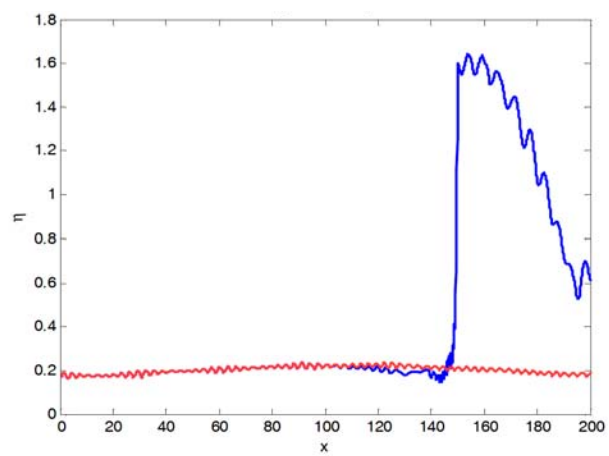

Figure 7. Comparison between MTA for flat bottom (red) and varying bottom (blue).

6. Conclusion

In this paper, we discussed the propagation of the bichromatic wave over varying bottom using AB equation as a governing equation. The paper shows that the maximal amplitude of the wave and its amplification are depended so much by depth of the tank. For the case that we take here, the maximal amplitude of the wave is happened in the lower bottom and more than 10 times compared with the constant depth.

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