Advances and Applications in Fluid Mechanics

http://www.pphmj.com
http://dx.doi.org/10.17654/FM021010027
Volume 21, Number 1, 2018, Pages 27-40 **ISSN: 0973-4686**

MONOCHROMATIC WAVE PROPAGATION IN KP-TYPE EQUATION

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Abstract

KP-type equation is known as a wave equation with two spatial dimensions. The equation was derived in 1970 by Kadomtsev-Petviashvili. In this paper, we discuss the monochromatic wave propagation in KP-type equation that contains exact dispersion relation and nonlinearity in the *x*-direction. AB equation is taken as the nonlinear wave equation contained in that KP-type. The equation is revised version of the KdV equation for wave about finite depth and in certain approximation it becomes the KdV equation. The third order asymptotic method is applied for solving the KP-type equation by choosing monochromatic wave as a signal input. Comparison between the dispersion relation of KP_{AB} and the other KP type is discussed to analyze the suitability of wave number of the KP. We also simulate the monochromatic wave propagation using the solution of the KP.

Received: June 3, 2016; Accepted: July 9, 2016

2010 Mathematics Subject Classification: 34E05, 74J30, 76B15.

Keywords and phrases: KP-type equation, AB-equation, third order asymptotic method, monochromatic wave.

Communicated by Shahrdad G. Sajjadi

1. Introduction

KP-type equation was derived in 1997 by Kadomtsev-Petviashvili as a generalization of Korteweg de Vries equation (KdV). The difference between KP and KdV equation is in the spatial dimension [7]. KP equation has two spatial dimensions, so sometimes we call for KP type equation to be a multidirectional wave equation. In this paper, we discuss the monochromatic wave propagation in KP-type equation that contains exact dispersion relation and nonlinearity in the *x*-direction.

AB equation is a new type of KdV equation. The equation has been presented in 2007. The model is an improvement of the KdV equation and can be interpreted as a higher order KdV equation for waves above a finite depth and under certain approximation it becomes the KdV equation [1, 2]. The model has exact dispersion properties and the nonlinear terms of the model is also improved to include the effects of short wave interactions. The model is given by

$$\partial_t \eta = -\sqrt{g}A\left[\eta + \frac{1}{2}A(\eta A\eta) - \frac{1}{4}(A\eta)^2 + \frac{1}{2}B(\eta B\eta) + \frac{1}{4}(B\eta)^2\right], \tag{1}$$

where η represents elevation of the wave, $A = \frac{\partial_x C}{\sqrt{g}}$ and $B = \sqrt{g}C^{-1}$ are pseudo-differential operators with Fourier's symbol $\hat{C}(k) = \frac{\Omega(k)}{k}$, $\Omega(k) = c_0 k\sqrt{\frac{\tanh(kh_0)}{kh_0}}$, $c_0 = \sqrt{gh_0}$, g and h_0 are acceleration gravitation and water depth, respectively.

First, we begin with derivation of KP type equation from the simplest second order wave equation:

$$\partial_t^2 \eta = c_0^2(\partial_x^2 + \partial_y^2)\eta, \tag{2}$$

with $c_0 = \sqrt{gh}$, where the dispersion relation is

$$\omega^2 = c_0^2(k_x^2 + k_y^2), \tag{3}$$

which assigns the frequency of wave ω with wave numbers of a harmonic wave in the directions *x* and *y*. The unidirectional wave with only direction *x* is given by

$$\partial_t^2 \eta = c_0^2 \partial_x^2 \eta. \tag{4}$$

Equation (4) can be rewritten as

$$(\partial_t - c_0 \partial_x)(\partial_t + c_0 \partial_x)\eta = 0, \tag{5}$$

equation (5) represents wave equation that propagates towards the left and right directions. The right direction of the wave propagation in (5) is written by

$$\partial_t \eta + c_0 \partial_x \eta = 0. \tag{6}$$

The dispersion relation of (6) is $\omega = c_0 k_x$, while the multidirectional wave that propagates only to the direction *x* has $k_y << k_x$, therefore its dispersion relation is given by:

$$\begin{aligned} \omega &= c_0 k_x \sqrt{1 + (k_y / k_x)^2} \\ &\approx c_0 k_x \left(1 + \frac{1}{2} \frac{k_y^2}{k_x^2} \right) = c_0 \left(k_x + \frac{1}{2} \frac{k_y^2}{k_x} \right). \end{aligned} \tag{7}$$

Equation (7) can be written in differential equation form as

$$\partial_t \eta = -c_0 (\partial_x \eta + \partial_x^{-1} \partial_y^2 \eta). \tag{8}$$

We rewrite the equation (8) in more appealing way as

$$\partial_x [\partial_t \eta + c_0 \partial_x \eta] + \frac{c_0}{2} \partial_y^2 \eta = 0. \tag{9}$$

Equation (9), which in the literature is often called the *standard* of KP equation that includes the wave equation in *x* direction only as the simple wave equation $\partial_t \eta + c_0 \partial_x \eta = 0$.

In the same way, if we take the dispersion relation and the nonlinear wave equation of x direction as the classical KdV

$$\partial_t \eta + \partial_x \eta + \frac{1}{6} \partial_x^3 \eta + \frac{3}{2} \eta \partial_x \eta = 0, \tag{10}$$

we obtain the standard KP equation as follows:

$$\partial_x \left[\partial_t \eta + \partial_x \eta + \frac{1}{6} \partial_x^3 \eta + \frac{3}{2} \eta \partial_x \eta \right] + \frac{c_0}{2} \partial_y^2 \eta = 0. \tag{11}$$

While, if the nonlinear wave equation is taken as AB equation

$$\partial_t \eta = -\sqrt{g} A \left[\eta + \frac{1}{2} A(\eta A \eta) - \frac{1}{4} (A\eta)^2 + \frac{1}{2} B(\eta B \eta) + \frac{1}{4} (B\eta)^2 \right],$$

we obtain the standard KP equation as

$$\partial_x \left[\partial_t \eta + \sqrt{g} A \left[\eta + \frac{1}{2} A(\eta A \eta) - \frac{1}{4} (A\eta)^2 + \frac{1}{2} B(\eta B \eta) + \frac{1}{4} (B\eta)^2 \right] \right]$$

$$+ \frac{c_0}{2} \partial^2 \eta = 0. \tag{12}$$

We then denote the KP equation in (12) by $\mathrm{KP_{AB}}$. The derivation of $\mathrm{KP_{AB}}$ can also be seen in [3].

In this paper, we use equation (12) as a model of two spatial dimensions water wave and using the asymptotic method solve the equation and also study the wave propagation. The paper is organized as follows: In Section 2, we discuss the comparison of dispersion relation of $\mathrm{KP_{AB}}$ and KP with KdV, to see the relation between wave numbers k_x and k_y for each of the equations. The asymptotic method is used to solve the $\mathrm{KP_{AB}}$ and monochromatic wave as a signal input is discussed in Section 3. In Section 4, we discuss the simulation of $\mathrm{KP_{AB}}$ together with the contribution of each term in KP solution. Conclusion is provided in Section 5.

2. Dispersion Relation of KP-type Equation

In this section, we discuss the dispersion relation of KP type with KdV equation that is given in (11) and KP_{AB} equation as in (12). For KP type with KdV, the dispersion relation is obtained by taking the linear part of the equation

$$\partial_x\left[\partial_t\eta + \partial_x\eta + \frac{1}{6}\partial_x^3\eta\right] + \frac{c_0}{2}\partial_y^2\eta = 0. \tag{13}$$

We can rewrite the above equation as

$$\partial_x\partial_t\eta + \partial_x\partial_x\eta + \frac{1}{6}\partial_x^4\eta + \frac{c_0}{2}\partial_y^2\eta = 0. \tag{14}$$

By choosing a monochromatic wave $\eta = ae^{i(k_x x + k_y y - \omega t)} + c.c,$ we get

$$k_x\omega - k_x^2 + \frac{1}{6}k_x^4 - \frac{c_0}{2}k_y^2 = 0. \tag{15}$$

The dispersion relation is obtained from (15) as

$$\omega = k_x - \frac{1}{6}k_x^3 + \frac{c_0}{2}\frac{k_y^2}{k_x},$$

where $c_0 = \sqrt{gh_0}$.

In the same way for KP_{AB}, its dispersion relation is obtained as follows:

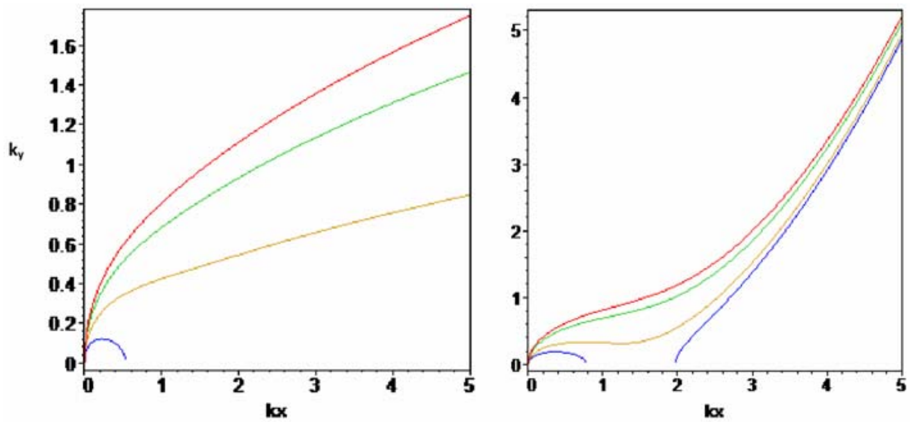
$$k_x\omega - \sqrt{g}k_x^2\frac{\Omega(k_x)}{\sqrt{g}k_x} - \frac{c_0}{2}k_y^2 = 0$$

$$k_x\omega = k_x\Omega(k_x) + \frac{c_0}{2}k_y^2,$$

so we get

$$\omega = \Omega(k_x) + \frac{c_0}{2}\frac{k_y^2}{k_x}. \tag{16}$$

The relation between wave numbers k_x and k_y for KP-type with KdV and KP_{AB} equation for various ω is given in Figure 1, where $\omega = 3.14$ (red), $\omega = 2.5$ (green), $\omega = 1.2$ (yellow), $\omega = 0.7$ (blue).



(a) (b)

Figure 1. The relation between wave numbers k_x and k_y for KP_{AB} (a) and KP type with KdV (b).

Figure 1 shows that wave numbers k_x and k_y in KP_{AB} satisfies $k_y << k_x$ for various ω as well as required in KP type equation [7], but for KP with KdV, the wave numbers k_x and k_y are satisfied only for $\omega \geq 5$.

3. Third Order Asymptotic Calculation of KP Equation

In this section, we discuss a solution of KP type equation using the third order asymptotic method. Mashuri et al. used the method for solving nonlinear water wave equation while studying the propagation of surface water wave based on modified KdV-type equation [4] and deriving Nonlinear Schrödinger equation (NLS_{AB}) in [5]. Also, the method is used for studying the bi-chromatics wave propagation over varying bottom [6]. We rewrite the KP_{AB} as

$$\partial_x\left[\partial_t\eta + \sqrt{g}A\left[\eta + \frac{1}{2}A(\eta A\eta) - \frac{1}{4}(A\eta)^2 + \frac{1}{2}B(\eta B\eta) + \frac{1}{4}(B\eta)^2\right]\right]$$

$$+ \frac{c_0}{2}\partial_y^2\eta = 0.$$

First, we take the solution of the equation as power series in ε until the third order:

$$\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)}. \tag{17}$$

By substituting (15) into (12), and collecting the polynomial terms in ε, ε^2 and ε^3, we get three partial differential equations as follows:

$$\partial_x(\partial_t\eta^{(1)} + \sqrt{g}A\eta^{(1)}) + \frac{c_0}{2}\partial_y^2\eta^{(1)} = 0, \tag{18}$$

$$\partial_x(\partial_t\eta^{(2)} + \sqrt{g}A\eta^{(2)}) + \frac{c_0}{2}\partial_y^2\eta^{(2)} = \partial_x\left(-\sqrt{g}A\left[\frac{1}{2}A(\eta^{(1)}A\eta^{(1)}) - \frac{1}{4}(A\eta^{(1)})^2\right.\right.$$

$$\left.\left. + \frac{1}{2}B(\eta^{(1)}B\eta^{(1)}) + \frac{1}{4}(B\eta^{(1)})^2\right]\right), \tag{19}$$

$$\partial_x(\partial_t\eta^{(3)} + \sqrt{g}A\eta^{(3)}) + \frac{c_0}{2}\partial_y^2\eta^{(3)}$$

$$= \partial_x\left(-\sqrt{g}A\left[\frac{1}{2}A(\eta^{(1)}A\eta^{(2)} + \eta^{(2)}A\eta^{(1)}) - \frac{1}{4}(A\eta^{(1)}A\eta^{(2)} + A\eta^{(2)}A\eta^{(1)})\right.\right.$$

$$\left.\left. + \frac{1}{2}B(\eta^{(1)}B\eta^{(2)} + \eta^{(2)}B\eta^{(1)}) + \frac{1}{4}(B\eta^{(1)}B\eta^{(2)} + B\eta^{(2)}B\eta^{(1)})\right]\right). \tag{20}$$

In the first order equation (18), the solution can be chosen monochromatic wave with amplitude a, frequency ω and wave numbers along x and y directions as k_x and k_y, respectively. We have

$$\eta^{(1)} = ae^{i\theta} + c.c, \tag{21}$$

with $\theta = k_x x + k_y y - \omega t$ and $c.c$ representing the complex conjugate.

Substituting (21) into (18), we obtain

$$\omega = \Omega(k_x) + \frac{c_0}{2}\frac{k_y^2}{k_x}, \tag{22}$$

with

$$\Omega(k_x) = c_0 k_x \sqrt{\frac{\tanh(k_x h_0)}{k_x h_0}}.$$

The relation between frequency and wave numbers in (22) is called the *exact dispersion relation* of KP.

By using asymptotic method for the wave equation, a resonance term appears in the third order term of the solution as a consequence of the nonlinearity. The resonance term is produced from interaction between the first and second order solutions. The resonant contribution has to be made to vanish in order to satisfy valid condition for asymptotic solution. To achieve this, we use Lindstedt-Poincare's method [8] by correcting the wave numbers and doing expansion of wave numbers. The method was also used by Mashuri et al. [4, 6] for the same reason. The wave number is given in the power series as follows:

$$k_x = k_x^{(0)} + \varepsilon k_x^{(1)} + \varepsilon^2 k_x^{(2)},$$

$$k_y = k_y^{(0)} + \varepsilon k_y^{(1)} + \varepsilon^2 k_y^{(2)}. \tag{23}$$

Therefore, the first order solution (18) is given by

$$\eta^{(1)} = ae^{i\theta} + c.c, \tag{24}$$

with $\theta = (k_x^{(0)} + \varepsilon k_x^{(1)} + \varepsilon^2 k_x^{(2)})x + (k_y^{(0)} + \varepsilon k_y^{(1)} + \varepsilon^2 k_y^{(2)})y - \omega t.$ The exact dispersion relation is obtained by substituting (24) into (18) as follows:

$$\omega = \Omega(k_x^{(0)}) + \frac{c_0}{2}\frac{k_y^{(0)^2}}{k_x^{(0)}}, \tag{25}$$

where

$$\Omega(k_x^{(0)}) = c_0 k_x^{(0)} \sqrt{\frac{\tanh(k_x^{(0)} h_0)}{k_x^{(0)} h_0}}.$$

Substituting the first order solution into the second order equation (19) and using Taylor's expansion of A and C about $k_x^{(0)}$, we obtain

$$\begin{aligned}&\partial_x(\partial_t \eta^{(2)} + \sqrt{g} A \eta^{(2)}) + \frac{c_0}{2} \partial_y^2 \eta^{(2)} \\ &+ ia(-ik_x^{(1)}\omega + ik_x^{(0)}(k_x^{(0)} k_x^{(1)} C'(k_x^{(0)}) + k_x^{(1)} C(k_x^{(0)}))e^{i\theta}) \\ &- 2k_y^{(0)} k_y^{(1)} a e^{i\theta} = RHS_1,\end{aligned} \tag{26}$$

with RHS_1 as the interaction between the first order solution and the second order solution

$$\begin{aligned}RHS_1 = -2ik_x^{(0)} a^2 \sqrt{g}\Big[&-\frac{1}{4} A^2(k_x^{(0)}) A(2k_x^{(0)}) + \frac{1}{2} A(k_x^{(0)}) A^2(2k_x^{(0)}) \\ &+ \frac{1}{4} B^2(k_x^{(0)}) A(2k_x^{(0)}) + \frac{1}{2} B(k_x^{(0)}) B(2k_x^{(0)}) A(2k_x^{(0)})\Big] e^{2i\theta}.\end{aligned}$$

From (26), we get $(k_x^{(1)}, k_y^{(1)}) = (0, 0)$ and the second order solution is chosen to be of the form

$$\eta^{(2)} = a_2 e^{2i\theta} + c.c. \tag{27}$$

Substituting $\eta^{(2)}$ into the second order equation (26), we obtain the coefficient of the second order solution as

$$a_2 = \frac{\alpha_2}{(2k_x^{(0)}(2\omega - 2k_x^{(0)} C(2k_x^{(0)})) - 2c_0 (k_y^{(0)})^2)},$$

with

$$\alpha_2 = -2ik_x^{(0)}a^2\sqrt{g}\Big[-\frac{1}{4}A^2(k_x^{(0)})A(2k_x^{(0)}) + \frac{1}{2}A(k_x^{(0)})A^2(2k_x^{(0)})$$

$$+\frac{1}{4}B^2(k_x^{(0)})A(2k_x^{(0)}) + \frac{1}{2}B(k_x^{(0)})B(2k_x^{(0)})A(2k_x^{(0)})\Big].$$

The third order solution can be found by substituting the first and the second order solution into equation (20) in the form:

$$\partial_x(\partial_t\eta^{(3)} + \sqrt{g}A\eta^{(3)}) + \frac{c_0}{2}\partial_y^2\eta^{(3)}$$

$$+ a\Big(k_x^{(2)}\omega - k_x^{(0)}[k_x^{(0)}k_x^{(2)}C'(k_x^{(0)}) + k_x^{(1)2}C'(k_x^{(0)}) + k_x^{(2)}C(k_x^{(0)})]$$

$$-\frac{c_0}{2}(k_y^{(1)2} + k_y^{(0)}k_y^{(2)})\Big)e^{i\theta}$$

$$+\Big(2k_x^{(1)}\omega - 2k_x^{(0)}[4k_x^{(0)}k_x^{(1)}C'(2k_x^{(0)}) + 2k_x^{(1)}C(2k_x^{(0)})]$$

$$-\frac{c_0}{2}4(2k_y^{(0)}k_y^{(1)})\Big)a_2e^{2i\theta} = \gamma_1e^{3i\theta} + \gamma_2e^{i\theta} + c.c,$$

with

$$\gamma_1 = -3ik_x^{(0)}\sqrt{g}\Big[-\frac{1}{2}aa_2A(k_x^{(0)})A(2k_x^{(0)}) + \frac{1}{2}aa_2(A(k_x^{(0)})$$

$$+ A(2k_x^{(0)}))A(3k_x^{(0)}) + \frac{1}{2}aa_2B(k_x^{(0)})B(2k_x^{(0)})$$

$$+\frac{1}{2}aa_2(B(2k_x^{(0)}) + B(k_x^{(0)}))B(3k_x^{(0)})\Big]A(3k_x^{(0)})e^{3i\theta},$$

$$\gamma_2 = -ik_x^{(0)}\sqrt{g}\Big[\frac{1}{2}aa_2A(k_x^{(0)})A(2k_x^{(0)}) + \frac{1}{2}aa_2(A(2k_x^{(0)}) - A(k_x^{(0)}))A(k_x^{(0)})$$

$$+\frac{1}{2}aa_2B(k_x^{(0)})B(2k_x^{(0)}) + \frac{1}{2}aa_2(B(2k_x^{(0)}) + B(k_x^{(0)}))B(k_x^{(0)})\Big]$$

$$\cdot A(k_x^{(0)})e^{i\theta}.$$

Since $k = (k_x^{(1)}, k_y^{(1)}) = (0, 0)$, we get the wave number corrections $k_x^{(2)}$ and $k_y^{(2)}$ which satisfy

$$a\left[k_x^{(2)}(\omega - k_x^{(0)}[k_x^{(0)}C'(k_x^{(0)}) + C(k_x^{(0)})]) - \frac{c_0}{2}k_y^{(0)}k_y^{(2)}\right] = \gamma_2. \tag{28}$$

Therefore, the wave number of the solution can be written as

$$k = (k_x, k_y) = (k_x^{(0)}, k_y^{(0)}) + (k_x^{(2)}, k_y^{(2)}).$$

The third order equation then becomes

$$\partial_x(\partial_t\eta^{(3)} + \sqrt{g}A\eta^{(3)}) + \frac{c_0}{2}\partial_y^2\eta^{(3)} = \gamma_1 e^{3i\theta} + c.c. \tag{29}$$

The third order contribution to the solution is taken as

$$\eta^{(3)} = a_3 e^{3i\theta} + c.c. \tag{30}$$

Substituting (30) into (29), we find the coefficient of the third order solution as

$$a_3 = \frac{\gamma_1}{a\left(3ik_x^{(0)}(-3i\omega + \sqrt{g}A(3k_x^{(0)}) - \frac{9c_0}{2}(k_y^{(0)})^2\right)}.$$

The solution of the KP_{AB} is given by

$$\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)},$$

where $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}$ are given in (24), (27) and (30), respectively.

4. Monochromatic Evolution

In this section, we simulate the KP_{AB} with monochromatic wave that discussed before as a signal input. For the simulation, the parameters of the wave are taken to be $a = 0.05\text{m}$; $h = 5\text{m}$; $g = 9.8\text{m/s}^2$; $k_x^{(0)} = 3/\text{m}$; $k_y^{(0)} = 1/\text{m}$; $k_y^{(2)} = 0/\text{m}$.

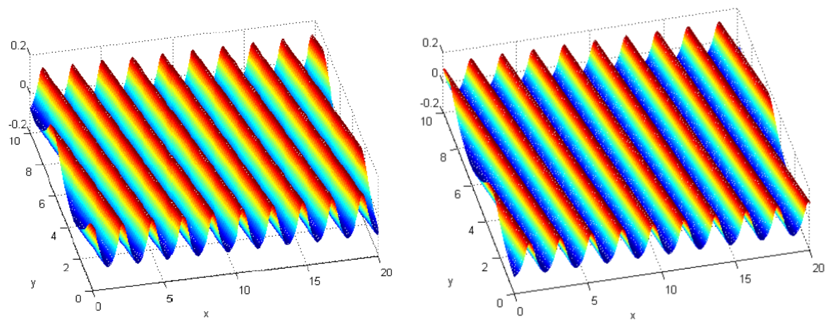
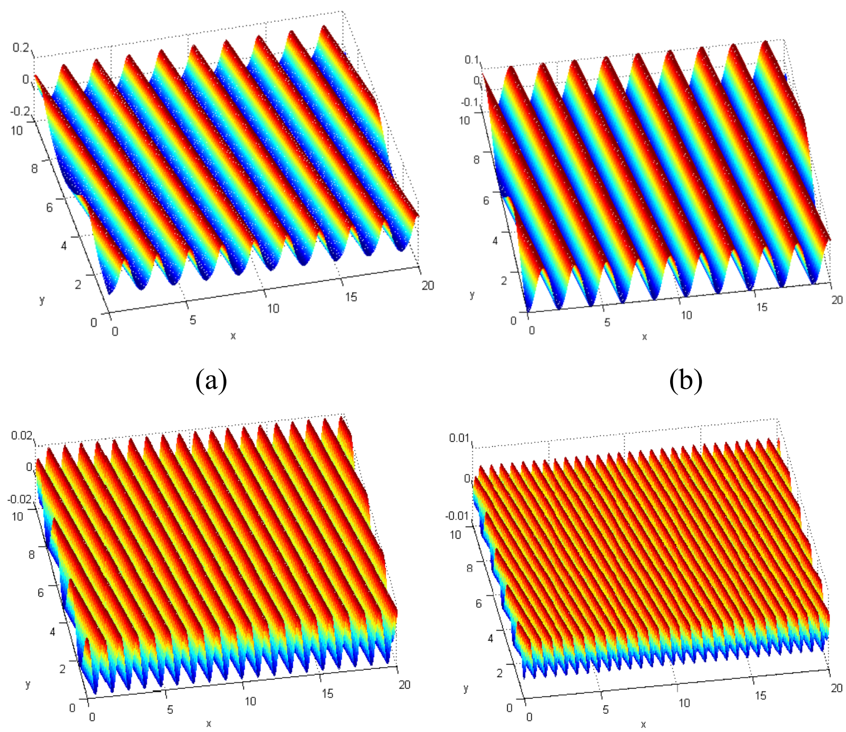


Figure 2. Wave profile at $t = 0$ (left) and $t = 10$ (right) with monochromatic as signal input.

The wave profiles and the contribution of each solution at $t = 0$ are given below.

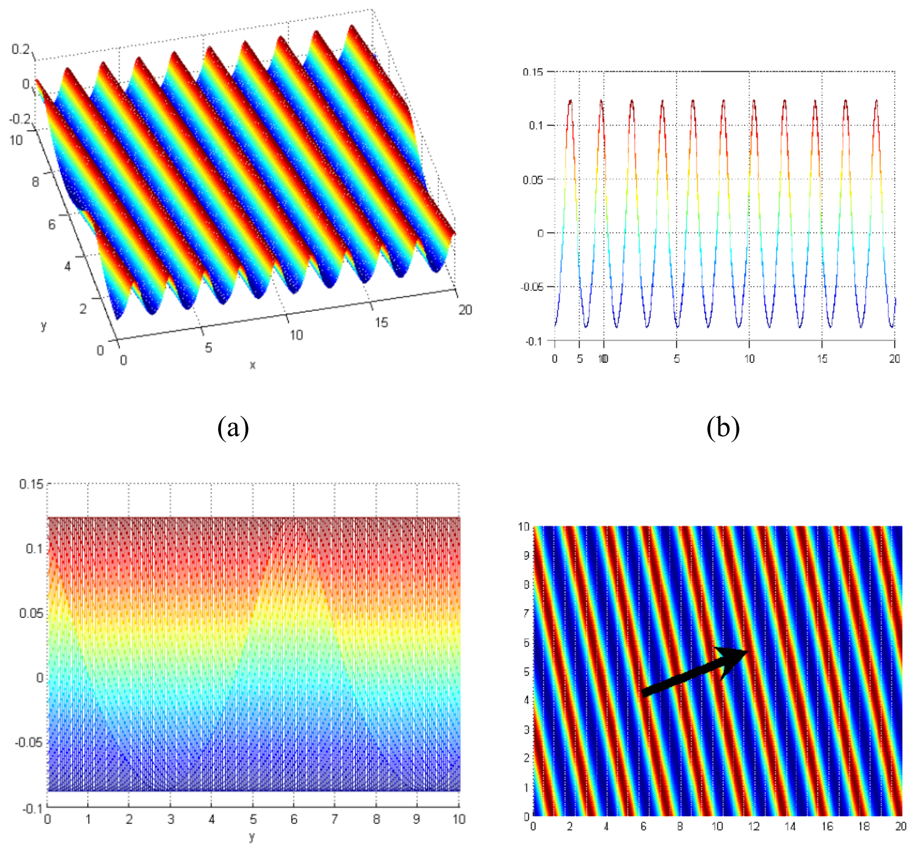


(a) (b)

(c) (d)

Figure 3. The profile of wave at $t = 0$ (a) with the first (b), second (c) and third (d) order contribution of the solution of KP.

While the wave profiles from each side are shown in Figure 4. Figure 4(a) shows wave profile at $t = 10$, (b) and (c) show wave profile in x-side and y-side, respectively, and the last figure (d) shows the direction of wave propagation.



(a) (b)

(c) (d)

Figure 4. Wave profile at $t = 0$ (a), looked from x side (b) and y side (c), the direction of wave propagation (d).

5. Conclusion and Remark

In this paper, we studied the monochromatic wave propagation using KP-type equation as wave model. By using in the same way when we derive

KP for the simplest wave equation, KP_{AB} can be derived including the AB equation as a wave equation in the *x*-direction. By using the dispersion relation, KP_{AB} is more satisfactory for the requirement of the wave number of KP- equation compared with KdV. Two wave numbers correction that we got from the third order solution depend on each other. Also, we simulated the monochromatic wave propagation for certain parameters of wave numbers and amplitude. The second and third order solutions give contribution of the first order solution depending on the parameters that we choose. But for monochromatic wave, the parameters do not make so significant effect to the first order solution.

Acknowledgments

We acknowledge the financial support by DIKTI bearing the contract number of DIPA-023.04.1.673453/2015. Also, we thank the referees for their careful reading of the manuscript and valuable advice.

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