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TRAVELING OF NONLINEAR BI-CHROMATIC WAVE IN THE HYDRODYNAMIC LABORATORY BASED ON KP-TYPE EQUATION

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Abstract

In this paper, we focus on investigating the traveling of bi-chromatic wave in the hydrodynamic laboratory. For the purpose, we use *KP*-type equation as a wave model. The equation is known as a water wave equation with two spatial dimensions proposed firstly by Kadomtsev-Petviashvili. Different from the other KP, in this paper, we propose the KP-type that contains the AB equation as a unidirectional equation. AB equation is an improvement of Korteweg de Vries (KdV) equation. The AB equation can be interpreted as a higher order KdV for wave about finite depth and in certain approximation it becomes the KdV equation. The solution is obtained by using the asymptotic method up to the third order. We also use the bi-chromatic wave as a signal input in the solution. Then we simulate the bi-chromatic wave propagation in the KP.

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1. Introduction

KP-type equation is a water wave equation with two spatial dimensions but mainly propagates only in one direction. Firstly, the KP equation was derived by Kadomtsev-Petviashvili in 1970. The equation is a generalization of the Korteweg de Vries (KdV) equation for two spatial dimensions [3, 8]. The equation is also a well-known model for dispersive, weakly nonlinear and almost unidirectional waves [4]. In this paper, we concentrate on studying the traveling of bi-chromatic wave in KP-type equation that contains the AB equation as a unidirectional wave model. The equation is then called in this paper as KP_{AB}.

AB equation is a new type of KdV equation. The model is an improvement of the KdV equation and can be interpreted as a higher order KdV equation for wave above finite depth and in certain approximation it becomes the KdV equation [2, 5-7]. The model has exact dispersion properties and the nonlinear terms of the model are also improved to include the effects of short wave interactions.

We first study the solution of KP_{AB} using the third order asymptotic method and taking the bi-chromatic wave as a signal input. The bi-chromatic is a sum of two monochromatic waves with the same amplitude and different wave numbers and frequencies. We choose the bi-chromatic wave because in practical situation the dispersion and nonlinearity develop much more complex wave phenomena such as large envelop deformations and breaking as reported in [9, 10]. The paper is organized as follows. In Section 2, we discuss the water wave model that we use in the paper. The asymptotic method is applied to solve KP_{AB} and choosing bi-chromatic wave as a signal input is discussed in Section 3. In Section 4, we discuss the simulation of bi-chromatic wave propagation using KP_{AB} and the contribution of each term in KP solution. Conclusion is given in Section 5.

2. Mathematical Model Equation

The wave equation that describes wave propagation with two spatial dimensions is given as follows. We start from the simplest water wave equation in x-direction

$$\partial_t^2 \eta = c_0^2 \partial_x^2 \eta. \tag{1}$$

Equation (1) can be rewritten as

$$(\partial_t - c_0 \partial_x)(\partial_t + c_0 \partial_x)\eta = 0. \tag{2}$$

Equation (2) describes the water wave equation that propagates to the right and left in x-direction. For the right direction of the equation $\partial_t \eta + c_0 \partial_x \eta = 0,$ it gives the dispersion relation $\omega = c_0 k_x.$ Meanwhile for the wave propagating in multi direction with mainly in x-direction has $k_y << k_x,$ therefore the dispersion relation is given by:

$$\omega = c_0 k_x \sqrt{1 + (k_y / k_x)^2}$$

$$\approx c_0 k_x \left(1 + \frac{1}{2} \frac{k_y^2}{k_x^2} \right) = c_0 \left(k_x + \frac{1}{2} \frac{k_y^2}{k_x} \right). \tag{3}$$

In other way, in differential equation form, equation (3) can be rewritten as

$$\partial_t \eta = -c_0 (\partial_x \eta + \partial_x^{-1} \partial_y^2 \eta). \tag{4}$$

In more appealing way, equation (4) can be rewritten as

$$\partial_x [\partial_t \eta + c_0 \partial_x \eta] + \frac{c_0}{2} \partial_y^2 \eta = 0. \tag{5}$$

Equation (5) is known as the standard of KP equation including the unidirectional wave equation in x-direction only as the simplest wave equation $\partial_t \eta + c_0 \partial_x \eta = 0.$ Meanwhile the KP-type equation will be obtained as

$$\partial_x \left[\partial_t \eta + \partial_x \eta + \frac{1}{6} \partial_x^3 \eta + \frac{3}{2} \eta \partial_x \eta \right] + \frac{c_0}{2} \partial_y^2 \eta = 0, \tag{6}$$

if we take the wave equation in x-direction as the classical KdV equation

$$\partial_t \eta + \partial_x \eta + \frac{1}{6}\partial_x^3 \eta + \frac{3}{2}\eta \partial_x \eta = 0. \tag{7}$$

In the same way, modified KdV equation with exact dispersion relation proposed in [1] gives the KP equation as

$$\partial_x[\partial_t \eta + i\Omega(-i\partial_x)\eta + \sigma\partial_x \eta^2] + \frac{c_0}{2}\partial_y^2 \eta = 0, \tag{8}$$

where the wave equation in x-direction is given by

$$\partial_t \eta + i\Omega(-i\partial_x)\eta + \sigma\partial_x \eta^2 = 0. \tag{9}$$

In this paper, we discuss the KP-type equation using the AB equation in [2] as a unidirectional wave equation given by

$$\partial_x\left[\partial_t \eta + \sqrt{g}A\left[\eta + \frac{1}{2}A(\eta A\eta) - \frac{1}{4}(A\eta)^2 + \frac{1}{2}B(\eta B\eta) + \frac{1}{4}(B\eta)^2\right]\right]$$

$$+ \frac{c_0}{2}\partial_y^2 \eta = 0, \tag{10}$$

with the AB equation given as:

$$\partial_t \eta = -\sqrt{g}A\left[\eta + \frac{1}{2}A(\eta A\eta) - \frac{1}{4}(A\eta)^2 + \frac{1}{2}B(\eta B\eta) + \frac{1}{4}(B\eta)^2\right], \tag{11}$$

where η represents the elevation of wave, $A = \frac{\partial_x C}{\sqrt{g}}$ and $B = \sqrt{g}C^{-1}$ represent pseudo-differential operators with symbol $\hat{C}(k) = \frac{\Omega(k)}{k}$, $\Omega(k) = c_0 k\sqrt{\frac{\tanh(kh_0)}{kh_0}}$, $c_0 = \sqrt{gh_0}$ and g represents acceleration gravitation. We then denote the KP equation in (10) by KP_{AB}. The derivation of KP_{AB} also can be seen in [4].

3. Asymptotic Solution of KP with Input Bi-chromatic Wave

In this section, we discuss the solution of KP_{AB} equation using third order asymptotic method. The method is used also in [5-7], when they solved the KdV equation and the AB equation. By using the third order asymptotic method, the elevation η is expanded in power series in ε up to third order, and written as

$$\eta = \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)}. \tag{12}$$

Substituting (12) to the KP_{AB} (11) and collecting the terms ε, ε^2 and ε^3, we obtain three partial differential linear equations as follows:

$$\partial_x(\partial_t\eta^{(1)} + \sqrt{g}A\eta^{(1)}) + \frac{c_0}{2}\partial_y^2\eta^{(1)} = 0, \tag{13}$$

$$\partial_x(\partial_t\eta^{(2)} + \sqrt{g}A\eta^{(2)}) + \frac{c_0}{2}\partial_y^2\eta^{(2)}$$

$$= \partial_x\begin{pmatrix} -\sqrt{g}A\Big[\frac{1}{2}A(\eta^{(1)}A\eta^{(1)}) - \frac{1}{4}(A\eta^{(1)})^2 \\ +\frac{1}{2}B(\eta^{(1)}B\eta^{(1)}) + \frac{1}{4}(B\eta^{(1)})^2\Big] \end{pmatrix}, \tag{14}$$

$$\partial_x(\partial_t\eta^{(3)} + \sqrt{g}A\eta^{(3)}) + \frac{c_0}{2}\partial_y^2\eta^{(3)}$$

$$= \partial_x\Big(-\sqrt{g}A\Big[\frac{1}{2}A(\eta^{(1)}A\eta^{(2)} + \eta^{(2)}A\eta^{(1)})$$

$$-\frac{1}{4}(A\eta^{(1)}A\eta^{(2)} + A\eta^{(2)}A\eta^{(1)}) + \frac{1}{2}B(\eta^{(1)}B\eta^{(2)} + \eta^{(2)}B\eta^{(1)})$$

$$+\frac{1}{4}(B\eta^{(1)}B\eta^{(2)} + B\eta^{(2)}B\eta^{(1)})\Big]\Big). \tag{15}$$

In the first order equation (13), we choose bi-chromatic wave which is the sum of two monochromatic waves with amplitude a, frequencies ω_1, ω_{12} and wave numbers k_x and k_y, respectively, in x- and y-directions. We write the bi-chromatic in the form

$$\eta^{(1)} = a(e^{i\theta_1} + e^{i\theta_2}) + c.c. \tag{16}$$

where $\theta_1 = k_{1x}x + k_{1y}y - \omega_1 t$, $\theta_2 = k_{2x}x + k_{2y}y - \omega_2 t$ and *c.c.* represents the complex conjugate.

Substituting equation (16) into (13) obtains the relations between wave number k and frequency ω as follows:

$$\omega_1 = \Omega(k_{1x}) + \frac{c_0}{2}\frac{k_{1y}^2}{k_{1x}}, \tag{17}$$

$$\omega_2 = \Omega(k_{2x}) + \frac{c_0}{2}\frac{k_{2y}^2}{k_{2x}}, \tag{18}$$

where

$$\Omega(k_{1x}) = c_0 k_{1x}\sqrt{\frac{\tanh(k_{1x}h_0)}{k_{1x}h_0}},$$

$$\Omega(k_{2x}) = c_0 k_{2x}\sqrt{\frac{\tanh(k_{2x}h_0)}{k_{2x}h_0}}.$$

The relations in equations (17) and (18) are known as exact dispersion relation of the KP_{AB}.

By using the asymptotic method for solving the AB equation will produce the resonance term in the third order solution as reported in [5-7]. The same condition for the KP_{AB} in handling the problem, we can correct the other parameter of the wave. Here, we use Linstedt-Poincare method for handling it by expanding wave number k as a power series in ε as follows:

$$k_{jx} = k_{jx}^{(0)} + \varepsilon k_{jx}^{(1)} + \varepsilon^2 k_{jx}^{(2)}; \quad j = 1, 2. \tag{19}$$

The first order solution (16) will be given by

$$\eta^{(1)} = a(e^{i\theta_1} + e^{i\theta_2}) + c.c. \tag{20}$$

with $\theta_j = (k_{jx}^{(0)} + \varepsilon k_{jx}^{(1)} + \varepsilon^2 k_{jx}^{(2)})x + (k_{jy}^{(0)} + \varepsilon k_{jy}^{(1)} + \varepsilon^2 k_{jy}^{(2)})y - \omega_j t;\ j = 1, 2.$

The exact dispersion relations (17) and (18) will be obtained as follows:

$$\omega_j = \Omega(k_{jx}^{(0)}) + \frac{c_0}{2}\frac{k_{jy}^{(0)^2}}{k_{jx}^{(0)}}; \quad j = 1, 2, \tag{21}$$

where

$$\Omega(k_{jx}^{(0)}) = c_0 k_{jx}^{(0)} \sqrt{\frac{\tanh(k_{jx}^{(0)} h_0)}{k_{jx}^{(0)} h_0}}; \quad j = 1, 2. \tag{22}$$

Figure 1 shows the relation between $k_{1x}^{(0)}$ and $k_{1y}^{(0)}$ that is suitable for assumption of the KP_{AB} for various ω.

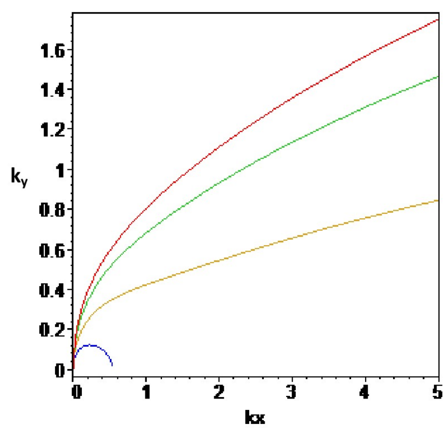


Figure 1. Relation of wave numbers in x-direction (k_{1x}) and y-direction (k_{1y}) for various ω.

Substituting the first order solution (20) into (14) and applying Taylor's expansion to the operators $\hat{A}$ and $\hat{C}$ about $k_x^{(0)}$ obtains the second order equation as follows:

$$\partial_x(\partial_t \eta^{(2)} + \sqrt{g} A \eta^{(2)}) + \frac{c_0}{2}\partial_y^2 \eta^{(2)} + a k_{1x}^{(1)}(\omega - i k_{1x}^{(0)}(k_{1x}^{(0)} C'(k_{1x}^{(0)}) + 2C(k_{1x}^{(0)})))e^{i\theta_1}$$

$$+ a k_{2x}^{(1)}(\omega - i k_{2x}^{(0)}(k_{2x}^{(0)} C'(k_{2x}^{(0)}) + 2C(k_{2x}^{(0)})))e^{i\theta_2}$$

$$- k_{1y}^{(0)} k_{1y}^{(1)} c_0 a e^{i\theta_1} - k_{2y}^{(0)} k_{2y}^{(1)} c_0 a e^{i\theta_2}$$

$$= \alpha_1 e^{2i\theta_1} + \alpha_2 e^{2i\theta_2} + \alpha_3 e^{i(\theta_1+\theta_2)} + \alpha_4 e^{i(\theta_1-\theta_2)} + c.c. \tag{23}$$

It should be noted that in equation (23), we drop the sign of hat (^) for simplicity. The right hand side of equation (23) is the interaction between the first order solution and itself. The coefficients of each term are given as follows:

$$\alpha_1 = -2ik_{1x}^{(0)} \sqrt{g} a^2 \Big[-\frac{1}{4} A^2(k_{1x}^{(0)}) A(2k_{1x}^{(0)}) + \frac{1}{2} A(k_{1x}^{(0)}) A^2(2k_{1x}^{(0)})$$

$$+ \frac{1}{4} B^2(k_{1x}^{(0)}) A(2k_{1x}^{(0)}) + \frac{1}{2} B(k_{1x}^{(0)}) B(2k_{1x}^{(0)}) A(2k_{1x}^{(0)}) \Big],$$

$$\alpha_2 = -2ik_{2x}^{(0)} \sqrt{g} a^2 \Big[-\frac{1}{4} A^2(k_{2x}^{(0)}) A(2k_{2x}^{(0)}) + \frac{1}{2} A(k_{2x}^{(0)}) A^2(2k_{2x}^{(0)})$$

$$+ \frac{1}{4} B^2(k_{2x}^{(0)}) A(2k_{2x}^{(0)}) + \frac{1}{2} B(k_{2x}^{(0)}) B(2k_{2x}^{(0)}) A(2k_{2x}^{(0)}) \Big],$$

$$\alpha_3 = -i(k_{1x}^{(0)} + k_{2x}^{(0)}) \sqrt{g} a^2 \Big[-\frac{1}{2} A(k_{1x}^{(0)}) A(k_{2x}^{(0)}) A(k_{1x}^{(0)} + k_{2x}^{(0)})$$

$$+ \frac{1}{2} (A(k_{1x}^{(0)}) + A(k_{2x}^{(0)})) A^2(k_{1x}^{(0)} + k_{2x}^{(0)})$$

$$+ \frac{1}{2} B(k_{1x}^{(0)}) B(k_{2x}^{(0)}) A(k_{1x}^{(0)} + k_{2x}^{(0)})$$

$$+ \frac{1}{2} (B(k_{1x}^{(0)}) + B(k_{2x}^{(0)})) B(k_{1x}^{(0)} + k_{2x}^{(0)}) A(k_{1x}^{(0)} + k_{2x}^{(0)}) \Big],$$

$$\alpha_4 = -i(k_{1x}^{(0)} - k_{2x}^{(0)}) \sqrt{g} a^2 \Big[\frac{1}{2} A(k_{1x}^{(0)}) A(k_{2x}^{(0)}) A(k_{1x}^{(0)} - k_{2x}^{(0)})$$

$$+ \frac{1}{2} (A(k_{1x}^{(0)}) - A(k_{2x}^{(0)})) A^2(k_{1x}^{(0)} k_{2x}^{(0)})$$

$$+\frac{1}{2}B(k_{1x}^{(0)})B(k_{2x}^{(0)})A(k_{1x}^{(0)}-k_{2x}^{(0)})$$

$$+\frac{1}{2}(B(k_{1x}^{(0)})+B(k_{2x}^{(0)}))B(k_{1x}^{(0)}-k_{2x}^{(0)})A(k_{1x}^{(0)}-k_{2x}^{(0)})\Big].$$

In this second order equation, we also get the first wave numbers correction given by:

$$(k_{jx}^{(1)},\ k_{jy}^{(1)})=(0,\ 0);\quad j=1,\ 2, \tag{24}$$

therefore, equation (24) will be

$$\partial_x(\partial_t\eta^{(2)}+\sqrt{g}A\eta^{(2)})+\frac{c_0}{2}\partial_y^2\eta^{(2)}$$

$$=\alpha_1e^{2i\theta_1}+\alpha_2e^{2i\theta_2}+\alpha_3e^{i(\theta_1+\theta_2)}+\alpha_4e^{i(\theta_1-\theta_2)}+c.c. \tag{25}$$

The second order solution from equation (25) is chosen to be of the form

$$\eta^{(2)}=a_{21}e^{2i\theta_1}+a_{22}e^{2i\theta_2}+a_{23}e^{i(\theta_1+\theta_2)}+a_{24}e^{i(\theta_1-\theta_2)}+c.c. \tag{26}$$

The coefficients of the second order solution are obtained by substituting equation (26) into (25) and given as follows:

$$a_{21}=\frac{\alpha_1}{(4k_{1x}^{(0)}\omega_1+2ik_{1x}^{(0)}\sqrt{g}A(2k_{1x}^{(0)})-2c_0(k_{1y}^{(0)})^2)},$$

$$a_{22}=\frac{\alpha_2}{(4k_{2x}^{(0)}\omega_2+2ik_{2x}^{(0)}\sqrt{g}A(2k_{2x}^{(0)})-2c_0(k_{2y}^{(0)})^2)},$$

$$a_{23}=\frac{\alpha_3}{(k_{1x}^{(0)}+k_{2x}^{(0)})((\omega_1+\omega_2)+i\sqrt{g}A(k_{1x}^{(0)}+k_{2x}^{(0)}))-\frac{c_0}{2}(k_{1y}^{(0)}+k_{2y}^{(0)})^2},$$

$$a_{24}=\frac{\alpha_4}{(k_{1x}^{(0)}-k_{2x}^{(0)})((\omega_1-\omega_2)+i\sqrt{g}A(k_{1x}^{(0)}-k_{2x}^{(0)}))-\frac{c_0}{2}(k_{1y}^{(0)}-k_{2y}^{(0)})^2}. \tag{27}$$

Meanwhile the third order solution can be found by substituting the first order solution (20) and the second order solution (26) into (15) as given by

$$\partial_x(\partial_t\eta^{(3)} + \sqrt{g}A\eta^{(3)}) + \frac{c_0}{2}\partial_y^2\eta^{(3)}$$

$$+\begin{pmatrix} k_{1x}^{(2)}\omega_1 a e^{i\theta_1} - 2k_{1x}^{(0)}k_{1x}^{(2)}C(k_{1x}^{(0)})ae^{i\theta_1} - k_{1x}^{(1)}k_{1x}^{(1)}C(k_{1x}^{(0)})ae^{i\theta_1} \\ - k_{1x}^{(0)}k_{1x}^{(0)}k_{1x}^{(2)}C'(k_{1x}^{(0)})ae^{i\theta_1} - \frac{c_0}{2}(k_{1y}^{(1)}k_{1y}^{(1)} + 2k_{1y}^{(0)}k_{1y}^{(2)})ae^{i\theta_1} \\ + k_{2x}^{(2)}\omega_2 a e^{i\theta_2} - 2k_{2x}^{(0)}k_{2x}^{(2)}C(k_{2x}^{(0)})ae^{i\theta_2} - k_{2x}^{(1)}k_{2x}^{(1)}C(k_{2x}^{(0)})ae^{i\theta_2} \\ - k_{2x}^{(0)}k_{2x}^{(0)}k_{2x}^{(2)}C'(k_{2x}^{(0)})ae^{i\theta_2} - \frac{c_0}{2}(k_{2y}^{(1)}k_{2y}^{(1)} + 2k_{2y}^{(0)}k_{2y}^{(2)})ae^{i\theta_2} \end{pmatrix}$$

$$+\begin{pmatrix} 4k_{1x}^{(1)}\omega_1 a_{21}e^{2i\theta_1} + 4k_{2x}^{(1)}\omega_2 a_{22}e^{2i\theta_2} + (k_{1x}^{(1)} + k_{1x}^{(1)})(\omega_1 + \omega_2)a_{23}e^{i(\theta_1+\theta_2)} \\ + (k_{1x}^{(1)} - k_{1x}^{(1)})(\omega_1 - \omega_2)a_{24}e^{i(\theta_1+\theta_2)} + (-8k_{1x}^{(0)}k_{1x}^{(1)}C(2k_{1x}^{(0)}) - 8k_{1x}^{(0)}k_{1x}^{(0)}k_{1x}^{(1)}C'(2k_{1x}^{(0)}))a_{21}e^{2i\theta_1} \\ + (-8k_{1x}^{(0)}k_{1x}^{(1)}C(2k_{1x}^{(0)}) - 8k_{1x}^{(0)}k_{1x}^{(0)}k_{1x}^{(1)}C'(2k_{1x}^{(0)}))a_{22}e^{2i\theta_2} \\ + (-2(k_{1x}^{(0)} + k_{2x}^{(0)})(k_{1x}^{(1)} + k_{2x}^{(1)})C(k_{1x}^{(0)} + k_{2x}^{(0)}) - (k_{1x}^{(0)} + k_{2x}^{(0)})^2(k_{1x}^{(1)} + k_{2x}^{(1)})C'(k_{1x}^{(0)} + k_{2x}^{(0)}))a_{23}e^{2i\theta_1} \\ + (-2(k_{1x}^{(0)} - k_{2x}^{(0)})(k_{1x}^{(1)} - k_{2x}^{(1)})C(k_{1x}^{(0)} - k_{2x}^{(0)}) - (k_{1x}^{(0)} - k_{2x}^{(0)})^2(k_{1x}^{(1)} - k_{2x}^{(1)})C'(k_{1x}^{(0)} - k_{2x}^{(0)}))a_{24}e^{2i\theta_1} \\ - 4c_0k_{1y}^{(0)}k_{1y}^{(1)}a_{21}e^{2i\theta_1} - 4c_0k_{2y}^{(0)}k_{2y}^{(1)}a_{22}e^{2i\theta_2} - c_0(k_{1y}^{(0)} + k_{2y}^{(0)})(k_{1y}^{(1)} + k_{2y}^{(1)})a_{23}e^{i(\theta_1+\theta_2)} \\ - c_0(k_{1y}^{(0)} - k_{2y}^{(0)})(k_{1y}^{(1)} - k_{2y}^{(1)})a_{24}e^{i(\theta_1-\theta_2)} \end{pmatrix} + c.c.$$

$$= \beta_1 e^{3i\theta_+} + \beta_2 e^{3i\theta_-} + \beta_3 e^{i(2\theta_+ + \theta_-)} + \beta_4 e^{i(2\theta_- + \theta_+)} + \beta_5 e^{i(2\theta_+ - \theta_-)}$$

$$+ \beta_6 e^{i(2\theta_- - \theta_+)} + \beta_7 e^{i\theta_+} + \beta_8 e^{i\theta_-} + c.c.$$

Since $(k_{1x}^{(1)}, k_{1y}^{(1)}) = (0, 0)$ and $(k_{2x}^{(1)}, k_{2y}^{(1)}) = (0, 0)$, we get

$$\partial_x(\partial_t\eta^{(3)} + \sqrt{g}A\eta^{(3)}) + \frac{c_0}{2}\partial_y^2\eta^{(3)}$$

$$+\begin{pmatrix} (k_{1x}^{(2)}(\omega_1 - 2k_{1x}^{(0)}C(k_{1x}^{(0)}) - k_{1x}^{(0)}k_{1x}^{(0)}C'(k_{1x}^{(0)})) - k_{1y}^{(2)}c_0k_{1y}^{(0)})ae^{i\theta_1} \\ + (k_{2x}^{(2)}(\omega_2 - 2k_{2x}^{(0)}C(k_{2x}^{(0)}) - k_{2x}^{(0)}k_{2x}^{(0)}C'(k_{2x}^{(0)})) - k_{2y}^{(2)}c_0k_{2y}^{(0)})ae^{i\theta_2} \end{pmatrix}$$

$$= \beta_1 e^{3i\theta_+} + \beta_2 e^{3i\theta_-} + \beta_3 e^{i(2\theta_+ + \theta_-)} + \beta_4 e^{i(2\theta_- + \theta_+)} + \beta_5 e^{i(2\theta_+ - \theta_-)}$$

$$+ \beta_6 e^{i(2\theta_- - \theta_+)} + \beta_7 e^{i\theta_+} + \beta_8 e^{i\theta_-} \tag{28}$$

with

$$\beta_1 = -3ik_{1x}^{(0)}\sqrt{g}\Big[-\frac{1}{2}aa_{21}A(k_{1x}^{(0)})A(2k_{1x}^{(0)}) + \frac{1}{2}aa_{21}(A(k_{1x}^{(0)}) + A(2k_{1x}^{(0)}))A(3k_{1x}^{(0)})$$

$$+\frac{1}{2}aa_{21}B(k_{1x}^{(0)})B(2k_{1x}^{(0)}) + \frac{1}{2}aa_{21}(B(k_{1x}^{(0)}) + B(2k_{1x}^{(0)}))B(3k_{1x}^{(0)})\Big]A(3k_{1x}^{(0)}),$$

$$\beta_2 = -3ik_{2x}^{(0)}\sqrt{g}\Big[-\frac{1}{2}aa_{22}A(k_{2x}^{(0)})A(2k_{2x}^{(0)}) + \frac{1}{2}aa_{22}(A(k_{2x}^{(0)}) + A(2k_{2x}^{(0)}))A(3k_{2x}^{(0)})$$

$$+\frac{1}{2}aa_{22}B(k_{2x}^{(0)})B(2k_{2x}^{(0)}) + \frac{1}{2}aa_{22}(B(k_{2x}^{(0)}) + B(2k_{2x}^{(0)}))B(3k_{2x}^{(0)})\Big]A(3k_{2x}^{(0)}),$$

$$\beta_3 = -i(2k_{1x}^{(0)} + k_{2x}^{(0)})\sqrt{g}\Big[-\frac{1}{2}aa_{23}A(k_{1x}^{(0)})A(k_{1x}^{(0)} + k_{2x}^{(0)}) - \frac{1}{2}aa_{21}A(k_{2x}^{(0)})A(2k_{1x}^{(0)})$$

$$+\Big(\frac{1}{2}aa_{23}(A(k_{1x}^{(0)}) + A(k_{1x}^{(0)} + k_{2x}^{(0)})) + \frac{1}{2}aa_{21}(A(k_{2x}^{(0)}) + A(2k_{1x}^{(0)}))\Big)A(2k_{1x}^{(0)} + k_{2x}^{(0)})$$

$$+\frac{1}{2}aa_{23}B(k_{1x}^{(0)})B(k_{1x}^{(0)} + k_{2x}^{(0)}) - \frac{1}{2}aa_{21}B(k_{2x}^{(0)})B(2k_{1x}^{(0)}) + \Big(\frac{1}{2}aa_{23}(B(k_{1x}^{(0)})$$

$$+B(k_{1x}^{(0)} + k_{2x}^{(0)})) + \frac{1}{2}aa_{21}(B(k_{2x}^{(0)}) + B(2k_{1x}^{(0)}))\Big)B(2k_{1x}^{(0)} + k_{2x}^{(0)})\Big]A(2k_{1x}^{(0)} + k_{2x}^{(0)}),$$

$$\beta_4 = -i(2k_{2x}^{(0)} + k_{1x}^{(0)})\sqrt{g}\Big[-\frac{1}{2}aa_{23}A(k_{2x}^{(0)})A(k_{1x}^{(0)} + k_{2x}^{(0)}) - \frac{1}{2}aa_{22}A(k_{1x}^{(0)})A(2k_{2x}^{(0)})$$

$$+\Big(\frac{1}{2}aa_{23}(A(k_{2x}^{(0)}) + A(k_{1x}^{(0)} + k_{2x}^{(0)})) + \frac{1}{2}aa_{22}(A(k_{1x}^{(0)}) + A(2k_{2x}^{(0)}))\Big)A(2k_{2x}^{(0)} + k_{1x}^{(0)})$$

$$+\frac{1}{2}aa_{23}B(k_{2x}^{(0)})B(k_{1x}^{(0)} + k_{2x}^{(0)}) - \frac{1}{2}aa_{22}B(k_{1x}^{(0)})B(2k_{2x}^{(0)}) + \Big(\frac{1}{2}aa_{23}(B(k_{2x}^{(0)})$$

$$+B(k_{1x}^{(0)} + k_{2x}^{(0)})) + \frac{1}{2}aa_{22}(B(k_{1x}^{(0)}) + B(2k_{2x}^{(0)}))\Big)B(2k_{2x}^{(0)} + k_{1x}^{(0)})\Big]A(2k_{2x}^{(0)} + k_{1x}^{(0)}),$$

$$\beta_5 = -i(2k_{1x}^{(0)} - k_{2x}^{(0)})\sqrt{g}\Big[-\frac{1}{2}aa_{24}A(k_{1x}^{(0)})A(k_{1x}^{(0)} - k_{2x}^{(0)}) + \frac{1}{2}aa_{21}A(k_{2x}^{(0)})A(2k_{1x}^{(0)})$$

$$+\Big(\frac{1}{2}aa_{24}(A(k_{1x}^{(0)}) + A(k_{1x}^{(0)} - k_{2x}^{(0)})) + \frac{1}{2}aa_{21}(A(2k_{1x}^{(0)}) - A(k_{2x}^{(0)}))\Big)A(2k_{1x}^{(0)} - k_{2x}^{(0)})$$

$$+\frac{1}{2}aa_{24}B(k_{1x}^{(0)})B(k_{1x}^{(0)}-k_{2x}^{(0)})-\frac{1}{2}aa_{21}B(k_{2x}^{(0)})B(2k_{1x}^{(0)})+\Big(\frac{1}{2}aa_{24}(B(k_{1x}^{(0)})$$

$$+B(k_{1x}^{(0)}-k_{2x}^{(0)}))+\frac{1}{2}aa_{21}(B(2k_{1x}^{(0)})-B(k_{2x}^{(0)}))\Big)B(2k_{1x}^{(0)}-k_{2x}^{(0)})\Big]A(2k_{1x}^{(0)}-k_{2x}^{(0)}),$$

$$\beta_6=-i(2k_{2x}^{(0)}-k_{1x}^{(0)})\sqrt{g}\Big[\frac{1}{2}aa_{24}A(k_{2x}^{(0)})A(k_{1x}^{(0)}-k_{2x}^{(0)})+\frac{1}{2}aa_{22}A(k_{1x}^{(0)})A(2k_{2x}^{(0)})$$

$$+\Big(\frac{1}{2}aa_{24}(A(k_{2x}^{(0)})-A(k_{1x}^{(0)}-k_{2x}^{(0)}))+\frac{1}{2}aa_{22}(A(2k_{2x}^{(0)})-A(k_{1x}^{(0)}))\Big)A(2k_{2x}^{(0)}-k_{1x}^{(0)})$$

$$+\frac{1}{2}aa_{24}B(k_{2x}^{(0)})B(k_{1x}^{(0)}-k_{2x}^{(0)})+\frac{1}{2}aa_{22}B(k_{1x}^{(0)})B(2k_{2x}^{(0)})+\Big(\frac{1}{2}aa_{24}(B(k_{2x}^{(0)})$$

$$+B(k_{1x}^{(0)}+k_{2x}^{(0)}))+\frac{1}{2}aa_{22}(B(k_{1x}^{(0)})+B(2k_{2x}^{(0)}))\Big)B(2k_{2x}^{(0)}-k_{1x}^{(0)})\Big]A(2k_{2x}^{(0)}-k_{1x}^{(0)}),$$

$$\beta_7=-ik_{1x}^{(0)}\sqrt{g}\Big[\frac{1}{2}aa_{21}A(k_{1x}^{(0)})A(2k_{1x}^{(0)})-\frac{1}{2}aa_{24}A(k_{2x}^{(0)})A(k_{1x}^{(0)}-k_{2x}^{(0)})$$

$$+\frac{1}{2}aa_{23}A(k_{2x}^{(0)})A(k_{1x}^{(0)}+k_{2x}^{(0)})$$

$$+\Big(\frac{1}{2}aa_{21}(A(2k_{1x}^{(0)})-A(k_{1x}^{(0)}))+\frac{1}{2}aa_{24}(A(k_{1x}^{(0)}-k_{2x}^{(0)})+A(k_{2x}^{(0)}))$$

$$+\frac{1}{2}aa_{23}(A(k_{1x}^{(0)}+k_{2x}^{(0)})-A(k_{2x}^{(0)}))\Big)A(k_{1x}^{(0)})$$

$$+\frac{1}{2}aa_{21}B(k_{1x}^{(0)})B(2k_{1x}^{(0)})+\frac{1}{2}aa_{24}B(k_{2x}^{(0)})B(k_{1x}^{(0)}-k_{2x}^{(0)})$$

$$+\frac{1}{2}aa_{23}B(k_{2x}^{(0)})B(k_{1x}^{(0)}+k_{2x}^{(0)})+\Big(\frac{1}{2}aa_{21}(B(2k_{1x}^{(0)})+B(k_{1x}^{(0)}))$$

$$+\frac{1}{2}aa_{24}(B(k_{1x}^{(0)}-k_{2x}^{(0)})+B(k_{2x}^{(0)}))$$

$$+\frac{1}{2}aa_{23}(B(k_{1x}^{(0)}+k_{2x}^{(0)})+B(k_{2x}^{(0)}))\Big)B(k_{1x}^{(0)})\Big]A(k_{1x}^{(0)}),$$

$$\beta_8 = -ik_{2x}^{(0)}\sqrt{g}\Big[\frac{1}{2}aa_{22}A(k_{2x}^{(0)})A(2k_{2x}^{(0)}) + \frac{1}{2}aa_{24}A(k_{1x}^{(0)})A(k_{1x}^{(0)} - k_{2x}^{(0)})$$

$$+ \frac{1}{2}aa_{23}A(k_{1x}^{(0)})A(k_{1x}^{(0)} + k_{2x}^{(0)}) + \Big(\frac{1}{2}aa_{22}(A(2k_{2x}^{(0)}) - A(k_{2x}^{(0)}))$$

$$+ \frac{1}{2}aa_{24}(-A(k_{1x}^{(0)} - k_{2x}^{(0)}) + A(k_{1x}^{(0)})) + \frac{1}{2}aa_{23}(A(k_{1x}^{(0)} + k_{2x}^{(0)}) - A(k_{1x}^{(0)}))\Big)A(k_{1x}^{(0)})$$

$$+ \frac{1}{2}aa_{22}B(k_{2x}^{(0)})B(2k_{2x}^{(0)}) + \frac{1}{2}aa_{24}B(k_{1x}^{(0)})B(k_{1x}^{(0)} - k_{2x}^{(0)})$$

$$+ \frac{1}{2}aa_{23}B(k_{1x}^{(0)})B(k_{1x}^{(0)} + k_{2x}^{(0)})$$

$$+ \Big(\frac{1}{2}aa_{22}(B(2k_{2x}^{(0)}) + B(k_{2x}^{(0)})) + \frac{1}{2}aa_{24}(B(k_{1x}^{(0)} - k_{2x}^{(0)}) + B(k_{1x}^{(0)}))$$

$$+ \frac{1}{2}aa_{23}(B(k_{1x}^{(0)} + k_{2x}^{(0)}) + B(k_{1x}^{(0)}))\Big)B(k_{1x}^{(0)})\Big]A(k_{2x}^{(0)}).$$

From equation (28), we get the wave number correction $k_{1x}^{(2)}$ and $k_{1y}^{(2)}$ for the first monochromatic and $k_{2x}^{(2)}$ and $k_{2y}^{(2)}$ for the second monochromatic

$$a(k_{1x}^{(2)}(\omega_1 - 2k_{1x}^{(0)}C(k_{1x}^{(0)}) - k_{1x}^{(0)}k_{1x}^{(0)}C'(k_{1x}^{(0)})) - k_{1y}^{(2)}c_0k_{1y}^{(0)}) = \beta_7,$$

$$a(k_{2x}^{(2)}(\omega_2 - 2k_{2x}^{(0)}C(k_{2x}^{(0)}) - k_{2x}^{(0)}k_{2x}^{(0)}C'(k_{2x}^{(0)})) - k_{2y}^{(2)}c_0k_{2y}^{(0)}) = \beta_8. \tag{29}$$

Moreover, the total wave number of the bi-chromatic is obtained as

$$(k_{1x}, k_{1y}) = (k_{1x}^{(0)}, k_{1y}^{(0)}) + (k_{1x}^{(2)}, k_{1y}^{(2)}),$$

$$(k_{2x}, k_{2y}) = (k_{2x}^{(0)}, k_{2y}^{(0)}) + (k_{2x}^{(2)}, k_{2y}^{(2)}) \tag{30}$$

with $(k_{1x}^{(2)}, k_{1y}^{(2)})$ and $(k_{2x}^{(2)}, k_{2y}^{(2)})$ satisfying (29). Therefore, the third order equation (28) will be

$$\partial_x(\partial_t \eta^{(3)} + \sqrt{g} A \eta^{(3)}) + \frac{c_0}{2} \partial_y^2 \eta^{(3)}$$

$$= \beta_1 e^{3i\theta_1} + \beta_2 e^{3i\theta_2} + \beta_3 e^{i(2\theta_1+\theta_2)}$$

$$+ \beta_4 e^{i(2\theta_2+\theta_1)} + \beta_5 e^{i(2\theta_1-\theta_2)} + \beta_6 e^{i(2\theta_2-\theta_1)} + c.c. \tag{31}$$

The solution of equation (31) can be chosen as

$$\eta^{(3)} = a_{31} e^{3i\theta_+} + a_{32} e^{3i\theta_-} + a_{33} e^{i(2\theta_+ + \theta_-)} + a_{34} e^{i(2\theta_- + \theta_+)}$$

$$+ a_{35} e^{i(2\theta_+ - \theta_-)} + a_{36} e^{i(2\theta_- - \theta_+)} + c.c. \tag{32}$$

We get the coefficients of the second third order solution by substituting (32) into (31) in the form

$$a_{31} = \frac{\beta_1}{3k_{1x}^{(0)}(3\omega_1 + i\sqrt{g}A(3k_{1x}^{(0)})) - \frac{9c_0}{2}(k_{1y}^{(0)})^2},$$

$$a_{32} = \frac{\beta_2}{3ik_{2x}^{(0)}(-3i\omega_2 + \sqrt{g}A(3k_{2x}^{(0)})) - \frac{9c_0}{2}(k_{2y}^{(0)})^2},$$

$$a_{33} = \frac{\beta_3}{i(2k_{1x}^{(0)} + k_{2x}^{(0)})(-i(2\omega_1 + \omega_2) + \sqrt{g}A(2k_{1x}^{(0)} + k_{2x}^{(0)})) - \frac{c_0}{2}(2k_{1y}^{(0)} + k_{2y}^{(0)})^2},$$

$$a_{34} = \frac{\beta_4}{i(2k_{2x}^{(0)} + k_{1x}^{(0)})(-i(2\omega_2 + \omega_1) + \sqrt{g}A(2k_{2x}^{(0)} + k_{1x}^{(0)})) - \frac{c_0}{2}(2k_{2y}^{(0)} + k_{1y}^{(0)})^2},$$

$$a_{35} = \frac{\beta_5}{i(2k_{1x}^{(0)} - k_{2x}^{(0)})(-i(2\omega_1 - \omega_2) + \sqrt{g}A(2k_{1x}^{(0)} - k_{2x}^{(0)})) - \frac{c_0}{2}(2k_{1y}^{(0)} + k_{2y}^{(0)})^2},$$

$$a_{36} = \frac{\beta_6}{i(2k_{2x}^{(0)} - k_{1x}^{(0)})(-i(2\omega_2 - \omega_1) + \sqrt{g}A(2k_{2x}^{(0)} - k_{1x}^{(0)})) - \frac{c_0}{2}(2k_{2y}^{(0)} - k_{1y}^{(0)})^2}.$$

We summarize that the solution of the KP_{AB} using third order asymptotic solution with the bi-chromatic wave as a signal input is given by

$$\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)},$$

with $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$ given by equations (20), (28) and (36), respectively.

4. Bi-chromatic Wave Propagation in KP_{AB} Equation

In this section, we simulate the propagation of bi-chromatic wave in KP_{AB} for various time with the parameter amplitude $a = 0.1771\text{m}$, $k_{1x} = 1{,}2511/\text{m}$, $k_{1y} = 1{,}1617/\text{m}$ related to frequency $\omega = 3{,}14\,\text{rad/s}$.

For $t = 0$, we have the bichromatic wave in the source and also the direction of wave propagation can be seen in Figure 2.

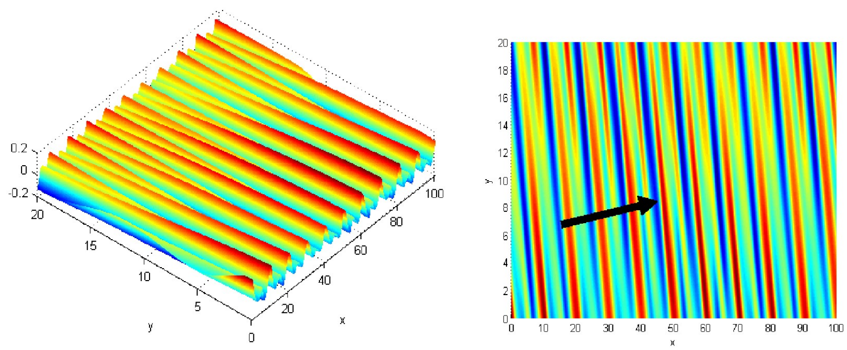
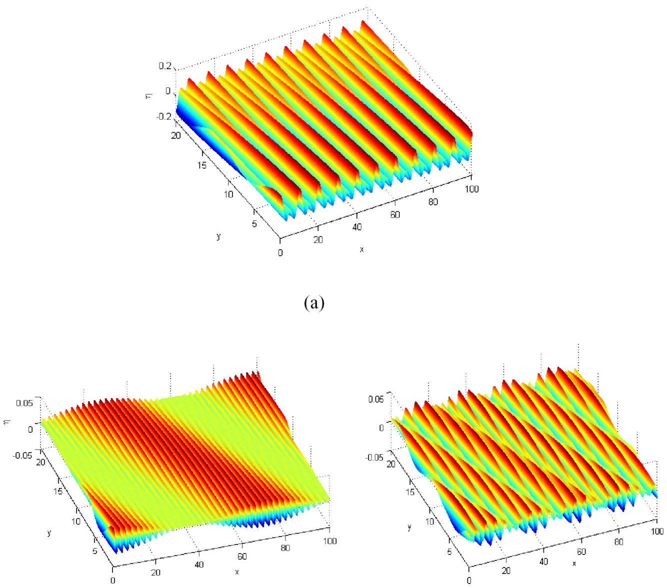


Figure 2. The bi-chromatic signal at $t = 0$ (left) and the direction of wave propagation (right).

The contribution of second order and third order solution of the KP_{AB} in the position $t = 0$ can be seen in Figure 3.

The second and third order solutions give contribution to the amplification of amplitude of the first order solution. We then simulate the bi-chromatic wave propagation at $t = 120\,\text{s}$. After 120 seconds, the amplitude of wave increases to 0.2m, and the signal at that time can be seen in Figure 4.



(a) (b) (c)

Figure 3. The second order solution $\eta^{(1)}$ (left) and the third order solution $\eta^{(2)}$ (right) that contribute to the first order solution $\eta^{(1)}$.

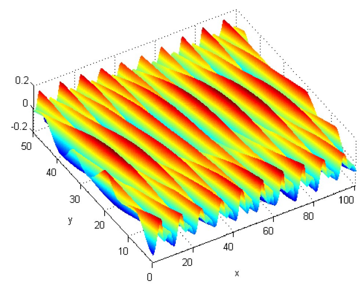


Figure 4. Signal of bi-chromatic wave at $t = 120\text{s}$ for $a = 0.1771\text{m}$, $k_{1x} = 1{,}2511/\text{m}$, $k_{1y} = 1{,}1617/\text{m}$.

5. Conclusion and Remark

In this paper, we studied the traveling of bi-chromatic wave in the hydrodynamic laboratory based on KP-type. By using third order asymptotic method, we obtained the third order asymptotic solution of the KP with bi-chromatic wave is taken as a signal input at the source. We also studied the bi-chromatic wave propagation using the solution of the equation. In the first order solution, we have result that the dispersion relation of KP_{AB} is satisfying the requirement of the wave number in the KP equation. Two wave number corrections that we obtained from the third order solution depend on each other. While the second and third order solutions give contribution to the first order solution and the parameters of the bi-chromatic are so significant effect to the first order solution. One of the effects can produce resonance term if the frequencies are close to each other.

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